Finite supernilpotent semigroups

Peter Mayr

PALS, February 4, 2025

Mathematics
UNIVERSITY OF COLORADO BOULDER

Question

What should

- abelian,
- nilpotent,
- solvable

mean for semigroups?

The general algebraic answer

Term condition commutator: a language independent commutator $[\alpha, \beta]$ for congruences α, β of any algebra **A**.

$[M, N]$ for normal subgroups M, N of a group G.

Guiding idea:
A is abelian if $\{(a, a) : a \in A\}$ is the class of a congruence on $\mathbf{A} \times \mathbf{A}$.

A group G is abelian iff $\{(g, g) \ : \ g \in G\}$ is normal in $G \times G$.

Then for $1/0$ the total/trivial congruence on **A**:

- **A** is **abelian** if $[1, 1] = 0$. $\qquad G' = 1$
- **A** is 2-**solvable** if $[[1, 1], [1, 1]] = 0$. $\qquad [G', G'] = 1$
- **A** is **(right)** 2-**nilpotent** if $[[1, 1], 1] = 0$. $\qquad [[G, G], G] = 1$
- **A** is 2-**supernilpotent** if $[1, 1, 1] = 0$.

What it means for semigroups

	abelian	nilpotent	super-nilpotent	solvable	
general	Warne 1994				
band	rectangular $L \times R$				
inverse	group				
monoid	embeds in group		???	???	M 2019
S with 0	$S^2 = 0$		$S^n = 0$		Mudrinski, Radović 2023
Rees matrix semigroup $M(G, I, \Lambda, P)$	$G \times L \times R$	G nilpotent		G solvable	Mudrinski, Radović 2023
finite	nilpotent extension of $M(G, I, \Lambda, P)$				M 2024

- For a group G, sets I, Λ and $P \in G^{\Lambda \times I}$,

$$M(G, I, \Lambda, P) := I \times G \times \Lambda \quad \text{with} \quad (i, g, \lambda) * (j, h, \mu) := (i, gP_{\lambda,j}h, \mu).$$

- S is a nilpotent extension of M if $S^n \subseteq M$ for some $n \in \mathbb{N}$.

The binary commutator

Binary commutators, term condition

higher

Freese, McKenzie's (1987) generalization from normal subgroups:

A algebra — group G

α, β congruences on **A** — $\equiv_M, \equiv_N$ for normal subgroups M, N

$$M(\alpha,\beta) := \left\langle \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} c & d \\ c & d \end{bmatrix} : a \equiv_\alpha b,\ c \equiv_\beta d \right\rangle \leq \mathbf{A}^{2\times 2}$$

Definition

The **commutator** $[\alpha, \beta]$ is the smallest congruence δ of **A** such that

$$\forall \begin{bmatrix} u & v \\ w & x \end{bmatrix} \in M(\alpha,\beta):\ u \equiv_\delta v \Rightarrow w \equiv_\delta x$$

$[\equiv_M, \equiv_N]$ is $\equiv_{[M,N]}$

$[\alpha, \beta]$ is the smallest δ such that $\forall \begin{bmatrix} u & v \\ w & x \end{bmatrix} \in \left\langle \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} c & d \\ c & d \end{bmatrix} \mid a \equiv_\alpha b, c \equiv_\beta d \right\rangle$:

$u \equiv_\delta v \Rightarrow w \equiv_\delta x$

Example

For $(\mathbb{N}, +)$ every element of $M(1,1)$ has the form

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & d \\ c & d \end{bmatrix} \quad \text{for } a, b, c, d \in \mathbb{N}\,.$$

Hence $u = v$ implies $w = x$.
Thus $[1, 1] = 0$.

$(\mathbb{N}, +)$ is **abelian**.

$[\alpha, \beta]$ is the smallest δ such that $\forall \begin{bmatrix} u & v \\ w & x \end{bmatrix} \in \left\langle \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} c & d \\ c & d \end{bmatrix} \mid a \equiv_\alpha b, c \equiv_\beta d \right\rangle$:

$u \equiv_\delta v \Rightarrow w \equiv_\delta x$

Example

For the semilattice $\mathbf{S} := (\{0,1\}, \cdot)$

$$M(1_{\mathbf{S}}, 1_{\mathbf{S}}) \ni \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Equality in row 1 $\Rightarrow$ row 2 is in the commutator.

$0 = 0$ implies $0 \equiv_{[1_{\mathbf{S}}, 1_{\mathbf{S}}]} 1$.

Thus $[1_{\mathbf{S}}, 1_{\mathbf{S}}] = 1_{\mathbf{S}}$.

S is commutative but **not abelian**.

Lemma

In semilattices, $[\alpha, \beta] = \alpha \wedge \beta$.

Just like in groups

For any algebra **A**

1. $[.,.]$ is monotone in both arguments
2. $[\alpha, \beta] \leq \alpha \wedge \beta$ $\quad [M, N] \leq M \cap N$
3. $[\alpha|_B, \beta|_B] \leq [\alpha, \beta]|_B$ for any subalgebra B of **A**
   $[M \cap H, N \cap H] \leq [M, N] \cap H$

By 3. subalgebras of abelian algebras ($[1, 1] = 0$) are abelian.

Corollary

Solvable semigroups cannot contain non-trivial subsemilattices.

All is not well

In congruence modular varieties (groups, rings, loops, lattices,...)

4. $[\alpha, \beta] = [\beta, \alpha]$ $[M, N] = [N, M]$
5. $[\alpha, \beta \vee \gamma] = [\alpha, \beta] \vee [\alpha, \gamma]$ $[M, NL] = [M, N][M, L]$
6. $[\alpha/\gamma, \beta/\gamma] = [\alpha, \beta]/\gamma$ $[M/L, N/L] = [M, N]/L$

Warning: All of these may fail for semigroups.

Counter example for 6.

$(\mathbb{N}, +)$ is abelian.
For the ideal $I := \mathbb{N} \setminus \{0\}$, the Rees quotient $\mathbb{N} / I \cong (\{0, 1\}, \cdot)$ is not abelian.

Quotients of abelian algebras ($[1, 1] = 0$) may not be abelian.

Nilpotence

Nilpotence

If commutators are not symmetric, there are 2 lower central series:

- **A** is **right** *n*-**nilpotent** if $[1)^{n+1} := [\ldots[\underbrace{[1,1],1],\ldots,1}_{n+1}] = 0.$
- **A** is **left** *n*-**nilpotent** if $(1]^{n+1} := [\underbrace{1,\ldots,[1,[1,1}_{n+1}]]\ldots] = 0.$

For any algebra **A**

abelian = right 1-nilpotent = left 1-nilpotent

abelian $\Rightarrow$ *n*-nilpotent $\Rightarrow$ *n*-solvable

Nilpotent monoids

Theorem (M 2019)

For a monoid **S** and $n \in \mathbb{N}$ TFAE:

1. **S** is left and right n-nilpotent.
2. **S** is cancellative and right n-nilpotent.
3. **S** is cancellative and satisfies $q_n(x, y, \overline{z}) = q_n(y, x, \overline{z})$.
4. **S** embeds into an n-nilpotent group.

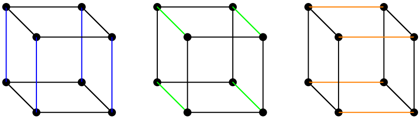
3.$\Leftrightarrow$ 4. by Mal'cev (1953), Neumann, Taylor (1963)

$$q_1(x, y, \overline{z}) := xy$$
$$q_{n+1}(x, y, \overline{z}) := q_n(x, y, \overline{z})\, z_n\, q_n(y, x, \overline{z})$$

Higher commutators

Higher commutators (Bulatov 2002)

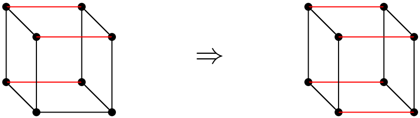
For congruences $\alpha_1, \alpha_2, \alpha_3$ of **A**, let $M(\alpha_1, \alpha_2, \alpha_3) \leq \mathbf{A}^{2^3}$ be generated by all cubes



Blue edges denote $(a_1, b_1) \in \alpha_1$, green $(a_2, b_2) \in \alpha_2$, orange $(a_3, b_3) \in \alpha_3$.

Definition

The ternary **higher commutator** $[\alpha_1, \alpha_2, \alpha_3]$ is the smallest congruence δ such that any element in $M(\alpha_1, \alpha_2, \alpha_3)$ satisfies the term condition



If the $2^2 - 1$ red edges on the left are in δ, then all are.

Basic properties of higher commutators

Let $k \in \mathbb{N}$.

- $[\alpha_1, \ldots, \alpha_k]$ is defined similarly via hypercubes in $\mathbf{A}^{2^k}$.
- In general $[\alpha_1, \ldots, \alpha_k]$ is not an iterated binary commutator.
  But for groups $[N_1, \ldots, N_k] = \prod_{\pi \in S_k} [..[N_{\pi(1)}, N_{\pi(2)}], \ldots, N_{\pi(k)}]$.

For any algebra $\mathbf{A}$

1. the higher commutator $[\ldots]$ is monotone in each argument
2. $[\alpha_1, \ldots, \alpha_k] \leq \alpha_1 \wedge \cdots \wedge \alpha_k$
3. $[\alpha_1|_B, \ldots, \alpha_k|_B] \leq [\alpha_1, \ldots, \alpha_k]|_B$ for any subalgebra B of $\mathbf{A}$
4. $[\alpha_1, \ldots, \alpha_k] \leq [\alpha_2, \ldots, \alpha_k]$

Supernilpotence

For any algebra **A**

$[1,1] \geq [1,1,1] \geq [1,1,1,1] \geq \dots$

A is *n*-**supernilpotent** if $[\underbrace{1,\dots,1}_{n+1}] = 0$.

- For groups, *n*-supernilpotent = *n*-nilpotent.
- In congruence modular varieties, a finite algebra is supernilpotent iff it is a direct product of nilpotent algebras of prime power order (Kearnes 1998).
- For finite algebras, supernilpotent ⇒ (left and right) nilpotent (Kearnes, Szendrei 2020).
- In Taylor varieties, supernilpotent ⇒ left nilpotent (Moorhead 2019) but not in general (Moore, Moorhead 2019)

Supernilpotent semigroups

Finite nilpotent semigroups are supernilpotent

Theorem (M 2024)

For a finite semigroup **S** TFAE:

1. **S** is left nilpotent.
2. **S** is right nilpotent.
3. **S** is supernilpotent.
4. **S** is a nilpotent extension of a Rees matrix semigroup $M(G, I, \Lambda, P)$ with nilpotent group G.

- Recall for a group G, sets I, Λ and $P \in G^{\Lambda \times I}$,

$$M(G, I, \Lambda, P) := I \times G \times \Lambda \quad \text{with} \quad (i, g, \lambda) * (j, h, \mu) := (i, gP_{\lambda,j}h, \mu).$$

- S is a nilpotent extension of M if $S^n \subseteq M$ for some $n \in \mathbb{N}$.

1., 2., 3. $\Rightarrow$ 4. follow from

- subalgebras of (super)nilpotent algebras are (super)nilpotent,
- facts on idempotents in finite semigroups.

The converse follow from

Theorem (M 2024)

Let **S** be a semigroup such that $S^d \cong M(G, I, \Lambda, P)$ and G is e-nilpotent. Then

1. **S** is $(2d + e)$-nilpotent,
2. **S** is $2de$-supernilpotent.

Proof 2: Let $S^d = M(G, I, \Lambda, P) =: R$ with G e-nilpotent

1. To show $[\underbrace{1, \ldots, 1}_{2de+1}] = 0$, consider $M(1, \ldots, 1) \leq \mathbf{S}^{2^{2de+1}}$ with generators constant on cosets of a hyperplane in 2^{2ed+1}. higher
2. $M(1, \ldots, 1) = \underbrace{\left\{ \begin{array}{c} \text{products of} \\ \geq d \text{ generators} \end{array} \right\}}_{=: U \leq R^{2^{2de+1}}} \cup \left\{ \begin{array}{c} \text{elements satisfying} \\ \text{TC with } = \end{array} \right\}$
3. $U = \left\langle \begin{array}{c} \text{products of} \\ d \text{ generators} \end{array} \right\rangle \leq \left\langle u \in R^{2^{2de+1}} : \begin{array}{c} u \text{ is constant on cosets of the} \\ \text{intersection of } d \text{ hyperplanes} \end{array} \right\rangle$
4. Multiplying 2 generators of U inserts a sandwich matrix over G that is constant on cosets of the intersection of $2d$ hyperplanes.
5. So the projection of U on its G-component is a subgroup of $V := \left\langle v \in G^{2^{2de+1}} : \begin{array}{c} v \text{ is constant on cosets of the} \\ \text{intersection of } 2d \text{ hyperplanes} \end{array} \right\rangle$
6. By group commutator calculus, V satisfies TC with $=$.
7. Hence $M(1, \ldots, 1)$ satisfies TC and $[\underbrace{1, \ldots, 1}_{2de+1}] = 0$. □

Open problems

Questions

1. Characterize nilpotent/solvable semigroups.
   - M 2019: Free monoids satisfy $[1,[1,1]] = 0$, hence are 2-solvable (but in general not right nilpotent)..
2. Determine central congruences $[\alpha, 1] = 0$, commutators $[\alpha, \beta]$
   - Kinyon, Stanovsky 2023: inverse semigroups
3. Is nilpotent = supernilpotent for semigroups?
   - M 2019: For monoids, 2-nilpotent = 2-supernilpotent.
   - There are finite examples with distinct nilpotency and supernilpotency degrees.