Filtered Boolean powers and their automorphism groups

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Outline

1. varieties generated by simple Mal'cev algebras
2. properties of filtered Boolean powers

1. Varieties

Primal algebras

A finite algebra **A** is **primal** if any $f\colon A^k \to A$ is a term function of **A**.

Example

- Boolean algebra of size 2
- $(\mathbb{Z}_p, +, \cdot, 0, 1)$ for p prime

Let $V := V(\mathbf{A})$ be the variety generated by a primal **A**.

- $V_{\text{fin}} = \{\mathbf{A}^k \mid k \in \mathbb{N}\}$
- $V =$ Boolean powers $\mathbf{A}^{\mathbf{B}}$ (Foster 1953)
- V is categorically equivalent to the variety of Boolean algebras (Hu 1969)

Functionally complete algebras

A finite algebra $\mathbf{A}$ is **functionally complete** if any $f\colon A^k \to A$ is a polynomial function of $\mathbf{A}$.

Example

- finite simple nonabelian groups
- finite fields

For Mal'cev algebras (having a ternary term $m(x, y, y) = m(y, y, x) = x$, e.g. (quasi)groups, rings, ...)

- functionally complete = simple nonabelian

Let $\mathbf{A}$ be a finite simple nonabelian Mal'cev algebra, $V := V(\mathbf{A})$, $W := V$(proper subalgebras of $\mathbf{A}$).

- $V_{\text{fin}} = \{\mathbf{A}^k \times \mathbf{C} \mid k \in \mathbb{N}, \mathbf{C} \in W_{\text{fin}}\}$

Finite direct powers

Lemma (Foster, Pixley, Werner)

Let **A** be a finite simple nonabelian Mal'cev algebra, $k, n \in \mathbb{N}$.
Then every homomorphism $h \colon \mathbf{A}^k \to \mathbf{A}^n$ is of the form

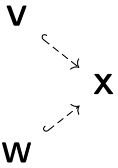
$$h(x_1, \ldots, x_k) = (\alpha_1(x_{i_1}), \ldots, \alpha_n(x_{i_n}))$$

for $\alpha_1, \ldots, \alpha_n$ automorphisms or endomorphisms to a 1-element subalgebra of **A**.

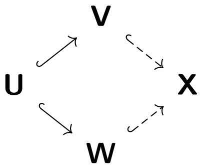
Finite direct powers are (essentially) a Fraïssé class

Let **A** be a finite simple nonabelian Mal'cev algebra.
$K := \{\mathbf{A}^k \mid k \in \mathbb{N}\}$ has

- the **joint embedding property (JEP)**,



- the **amalgamation property (AP)**,



- the **hereditary property (HP)** iff all proper subalgebras of **A** are trivial.

Note. In general $V(\mathbf{A})_{\mathrm{fin}}$ does not have AP.

Fraïssé limit

$K := \{\mathbf{A}^k \mid k \in \mathbb{N}\}$ for $\mathbf{A}$ a finite simple nonabelian Mal'cev algebra

Theorem (Fraïssé)

There exists a unique (up to isomorphism) countable algebra $\mathbf{D} =: \mathrm{Flim} K$, the **Fraïssé limit** of K, such that

1. every finitely generated subalgebra of $\mathbf{D}$ embeds into some element of K,
2. $\mathbf{D}$ is a direct limit of algebras in K,
3. every isomorphism between subalgebras of $\mathbf{D}$ that are in K extends to an automorphism of $\mathbf{D}$ (K-homogeneous).

Moreover, $\mathrm{Flim} K$ is ω-categorical.

Question

What is $\mathrm{Flim} K$ explicitly?

Filtered Boolean powers (Arens, Kaplansky 1948)

A algebra (with discrete topology),
B Boolean algebra with Stone space X.

$$\mathbf{A}^{\mathbf{B}} := \{f\colon X \to A \mid f \text{ continuous}\} \leq \mathbf{A}^X$$

is a **Boolean power**.

e is **idempotent** if $\{e\} \leq \mathbf{A}$.

For distinct $x_1, \ldots, x_n \in X$ and idempotents $e_1, \ldots, e_n$ in **A**,

$$(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n} := \{f \in \mathbf{A}^{\mathbf{B}} \mid f(x_i) = e_i \text{ for all } i\} \leq \mathbf{A}^X$$

is a **filtered Boolean power**.

The Fraïssé limit as filtered Boolean power

Theorem (M, Ruškuc 2023)

For a finite simple nonabelian Mal'cev algebra $\mathbf{A}$

$$\mathrm{Flim}(\mathbf{A}^k \mid k \in \mathbb{N}) \cong (\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n}$$

where

- $e_1, \ldots, e_n$ is the set of idempotents of $\mathbf{A}$,
- $\mathbf{B}$ is the countable atomless Boolean algebra with distinct $x_1, \ldots, x_n$ in the Cantor space X.

Proof.

Check that $(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n}$ satisfies the defining properties of the Fraïssé limit. □

Free algebras of countable rank in $V(\mathbf{A})$

Theorem (M, Ruškuc 2023; for groups Bryant, Groves 1991)

Let $\mathbf{A}$ be a finite simple nonabelian Mal'cev algebra, $V := V(\mathbf{A})$, θ minimal such that $\mathbf{F}_V(\omega)/\theta$ is in $W := V(\text{proper subalgebras of } \mathbf{A})$. Then each θ-class, which is a subalgebra of $\mathbf{F}_V(\omega)$, is isomorphic to

$$\mathrm{Flim}(\mathbf{A}^k \mid k \in \mathbb{N}) \cong (\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}.$$

Proof.

Consider $\mathbf{F}_V(\omega) \leq \mathbf{A}^{A^{\mathbb{N}}}$. □

Corollary (M, Ruškuc 2023)

For $\mathbf{A}$ a finite simple nonabelian group, loop, ring,

$$\mathbf{F}_V(\omega) \;\cong\; (\mathbf{A}^{\mathbf{B}})^{x_1}_{e_1} \rtimes \mathbf{F}_W(\omega).$$

2. Automorphisms of filtered Boolean powers

Largeness in permutation groups

Let M be a countable infinite structure, $G := \mathrm{Aut}M$.

- G is a topological (Polish) group under **pointwise convergence**: basic open sets are cosets of stabilizers of finite tuples over M

$$G_{m_1,\dots,m_k} := \{g \in G \mid g(m_i) = m_i \text{ for all } i \leq k\}.$$

- M has the **small index property (SIP)** if each $H \leq G$ of index $< 2^{\aleph_0}$ is open.
- G has **uncountable cofinality** if it is not a countable union of a chain of proper subgroups.
- G has the **Bergman property** if for each generating set $1 \in E = E^{-1}$ of G there exists $k \in \mathbb{N}$ such that $G = E^k$.
- G has **ample generics** if for each $n \in \mathbb{N}$ the conjugacy action of G on G^n has a comeager orbit (i.e. one containing the intersection of countably many dense open subsets of G^n).

Theorem (Kechris, Rosendal 2007)

1. Ample generics imply SIP and uniqueness of the Polish topology on $\mathrm{Aut} M$.
2. For ω-categorical M, ample generics imply uncountable cofinality and the Bergman property for $\mathrm{Aut} M$.

	SIP	uncountable cofinality	Bergman	ample generics
$\mathbb{N}$	Dixon Neumann Thomas '86	Macpherson Neumann '86	Bergman '06	Kechris Rosendal '07
random graph	Hodges Hodkinson Lascar Shelah '93	Hodges Hodkinson Lascar Shelah '93	Kechris Rosendal '07	Hrushovsky '92
$(\mathbb{Q}, \leq)$	Truss '89	Gourion '92	Droste Holland '05	no, Hodkinson
free group of rank ω	Bryant Evans '97	Bryant Evans '97	Tolstykh '07	Bryant Evans '97
Cantor space	Truss '87	Droste Göbel '05	Droste Göbel '05	Kwiatkowska '12
$(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n}$	M Ruškuc '23	M Ruškuc '23	M Ruškuc '23	???

ω-categorical filtered Boolean powers

A countable structure M is ω-**categorical** if its theory has a unique countable model (up to isomorphism).

Theorem

Let $\mathbf{A}$ be a finite simple nonabelian Mal'cev algebra,
$\mathbf{B}$ the countable atomless Boolean algebra.
Then any filtered Boolean power $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ is ω-categorical.

Proof.

By Macintyre, Rosenstein 1976

- the augmented Boolean algebra $(\mathbf{B}, x_1, \dots, x_n)$ is ω-categorical
- and hence $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ is.

□

Congruences

Congruences of a Boolean algebra $\mathbf{B}$ are determined by **filters** (the classes of 1).

The **equalizer** of $f, g \in \mathbf{A}^{\mathbf{B}}$ is

$$[[f = g]] := \{x \in X \mid f(x) = g(x)\}.$$

For a filter F on $\mathbf{B}$,

$$\theta_F := \{(f, g) \in \mathbf{A}^{\mathbf{B}} \mid [[f = g]] \in F\}$$

is a congruence of $\mathbf{A}^{\mathbf{B}}$.

Lemma (cf. Burris 1975)

Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra, $\mathbf{B}$ a Boolean algebra. Then

$$\mathrm{Con}(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n} = \{\theta_F \mid F \text{ is a filter contained in } \bigcap_{i=1}^{n} x_i\}.$$

Automorphism groups of filtered Boolean powers

Automorphisms of a Boolean algebra **B** correspond to **homeomorphisms** (continuous bijections) of its Stone space X,

$$\operatorname{Aut}\mathbf{B} \cong \operatorname{Homeo} X.$$

Theorem (M, Ruškuc 2023; for groups cf. Apps 1981)

Let **A** be a finite simple non-abelian Mal'cev algebra with idempotents $e_1, \ldots, e_n$ in distinct $\operatorname{Aut}\mathbf{A}$-orbits,
B the countable atomless Boolean algebra,
X the Cantor space with distinct $x_1, \ldots, x_n \in X$. Then

$$\operatorname{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n} \cong N \rtimes (\operatorname{Homeo} X)_{x_1,\ldots,x_n}$$

where N is isomorphic to the closure of $((\operatorname{Aut}\mathbf{A})^{\mathbf{B}})^{x_1,\ldots,x_n}_{1,\ldots,1}$.

SIP for expanded Boolean algebras

Theorem (M, Ruškuc 2023; for $n = 0$ Truss 1987)

Let $G := (\mathrm{Homeo} X)_{x_1,\ldots,x_n}$ for $x_1, \ldots, x_n$ in the Cantor space X, let $H \leq G$ such that $|G : H| < 2^{\aleph_0}$.
Then there exist clopens $b_1, \ldots, b_m$ partitioning X such that $G_{b_1,\ldots,b_m} \leq H$.

Proof.

Uses piecewise patching of homeomorphisms on clopens in $X \setminus \{x_1, \ldots, x_n\}$. □

SIP for filtered Boolean powers

Theorem (M, Ruškuc 2023, for groups Bryant, Evans 1997)

Let $\mathbf{A}$ be a finite simple nonabelian Mal'cev algebra,
$\mathbf{B}$ the countable atomless Boolean algebra.
Then any filtered Boolean power $(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ has SIP.

Proof.

- $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n} \cong N \rtimes C$ for $C := (\mathrm{Homeo}X)_{x_1,\dots,x_n}$.
- Let $H \leq \mathrm{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ have small index.
- Then $C \cap H$ has small index in C and contains $C_{b_1,\dots,b_m}$ for clopens $b_1,\dots,b_m$ partitioning X.
- $N \cap H$ has small index in N and is invariant under $C_{b_1,\dots,b_m}$.
- H contains the stabilizer of the finitely many functions that are constant on $b_1,\dots,b_m$.

□

Uncountable strong cofinality for expanded Boolean algebras

G has **uncountable strong cofinality** if G is not a countable union of proper subsets $U_1 \subseteq U_2 \subseteq \dots$ with $U_i = U_i^{-1}$ and $U_i^2 \subseteq U_{i+1}$ for all $i \in \mathbb{N}$.

Lemma (Droste, Göbel 2005)

uncountable strong cofinality
$\Leftrightarrow$ uncountable cofinality and Bergman property

Theorem (M, Ruškuc 2023; for $n = 0$ Droste, Göbel 2005)

Let X be the Cantor space, $x_1, \dots, x_n \in X$.
Then $(\mathrm{Homeo} X)_{x_1,\dots,x_n}$ has uncountable strong cofinality.

Proof.

Uses piecewise patching of homeomorphisms on clopens in $X \setminus \{x_1, \dots, x_n\}$. □

Uncountable strong cofinality for filtered Boolean powers

Theorem (M, Ruškuc 2023)

Let **A** be a finite simple nonabelian Mal'cev algebra,
B the countable atomless Boolean algebra.
Then for any filtered Boolean power $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ has uncountable strong cofinality.

Proof.

Uses the semidirect decomposition of $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ and the result on $(\mathrm{Homeo}X)_{x_1,\dots,x_n}$. □

Questions

Open

Let $\mathbf{A}$ be a finite simple nonabelian group (Mal'cev algebra).

Question

Is every $\mathbf{C}$ in $V(\mathbf{A})$ an extension of a filtered Boolean power of $\mathbf{A}$ by a group in V(proper subgroups of $\mathbf{A}$)?

True for countable $\mathbf{C}$.

Question (Bryant, Evans 1997)

Does the free group of countable rank in $V(\mathbf{A})$ have SIP (uncountable cofinality, Bergman property)?

Question

For $\mathbf{B}$ the countable atomless Boolean algebra, does $\mathrm{Aut}(\mathbf{A}^{\mathbf{B}})_{e_1}^{x_1}$ have ample generics?