Using Prover9 for research on ortholattices and locally integral involutive residuated po-monoids

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Outline

Part 1: Joint work with **J.B. Nation and Ralph Freese**, U. Hawaii

Ortholattice varieties and some equational bases

Part 2: Joint work with **Melissa Sugimoto** (CUNY),
José Gil-Férez and **Sid Lodhia** (Chapman)

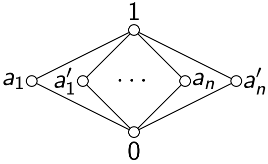
Plonka sums of locally integral involutive *po*-monoids

Ortholattices

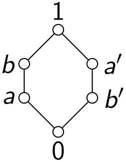
An **ortholattice** $(A, +, \cdot, ', 0, 1)$ is a lattice $(A, +, \cdot)$ with a unary **orthocomplement** $'$ that satisfies

$$x'' = x, \quad (x + y)' = x' \cdot y', \quad x \cdot x' = 0 \quad \text{and} \quad x + x' = 1.$$

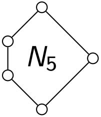
Examples: Boolean algebras, $MO_n =$



Benzene hexagon $H =$



Not an OL:



How does this look in Prover9 using Colab?

Go to `http://colab.research.google.com` and paste:

```
!pip install provers
!git clone https://github.com/jipsen/Prover9.git
from provers import *; execfile("/content/Prover9/Prover9.py")

OL=["(x+y)+z = x+(y+z)", "x+y = y+x", "x + x*y = x",
    "(x*y)*z = x*(y*z)", "x*y = y*x", "x*(x+y) = x",
    "x'' = x", "(x+y)' = x'*y'", "x*x' = 0", "x+x' = 1"]
L = p9(OL,[],100,0,[8])
```

The first 3 lines take 30 seconds (but only need to run once).

p9 calls Mace4 and Prover9 for 100 and 0 seconds respectively.

[8] means find all models up to cardinality 8

Prover9/Mace4 output

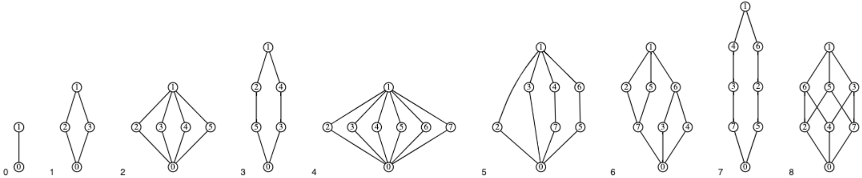
```
Number of nonisomorphic models of cardinality 2  is  1
No model of cardinality 3
Number of nonisomorphic models of cardinality 4  is  1
No model of cardinality 5
Number of nonisomorphic models of cardinality 6  is  2
No model of cardinality 7
Number of nonisomorphic models of cardinality 8  is  5
Fine spectrum:  [1, 1, 0, 1, 0, 2, 0, 5]
```

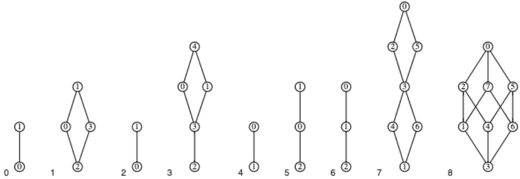
Mace is short for **Models and counterexamples**

See `https://www.cs.unm.edu/~mccune/prover9/manual/2009-11A/`

In Colab one can show diagrams of Mace4 output

```
L = L[2]+L[4]+L[6]+L[8]
show(L,"+")
show(Con(L))
```





Varieties of ortholattices

$\mathcal{OL} = \text{Mod}\{OL\}$ all **ortholattices**

$\mathcal{O} = \text{Mod}\{x = y\}$ all **one-element** ortholattices (relative to $\mathcal{OL}$)

$\mathcal{BA} = \text{Mod}\{x(y + z) = xy + xz\}$ all **Boolean algebras**

$\mathcal{MOL} = \text{Mod}\{(xz + y)z = xz + yz\}$ all **modular** ortholattices

$\mathcal{OML} = \text{Mod}\{x{+}x'(x{+}y) = x{+}y\}$ all **orthomodular** lattices

Some splittings in lattices (of varieties)

For a, b in a lattice L,

(a, b) is a **splitting pair** if $a \nleqslant b$ and $a \leqslant c$ or $c \leqslant b$ for all c

$\iff \uparrow a \cap \downarrow b = \emptyset$ and $\uparrow a \cup \downarrow b = L$.

E.g. $(\mathbb{V}(\mathbf{2}), \mathcal{O})$ is a splitting pair of varieties in $\Lambda_{\mathcal{L}}$ since a lattice has $\mathbf{2}$ as a sublattice $\iff$ it is nontrivial.

$(\mathbb{V}(N_5), \mathcal{ML})$ is a splitting pair since a lattice has N_5 as a sublattice $\iff$ it is nonmodular.

The lattice $\Lambda_{\mathcal{L}}$ of varieties of lattices

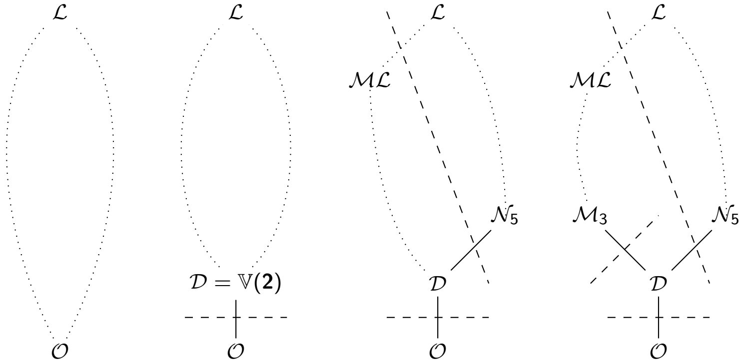


Figure: Splittings in the lattice $\Lambda_{\mathcal{L}}$ of lattice varieties

More about splittings for varieties (of ortholattices)

For a, b in a **complete** lattice L,

(a, b) is a **splitting pair** $\iff$ a is completely join prime
$\iff$ b is completely meet prime

$(\mathcal{U}, \mathcal{V})$ is a splitting pair of varieties
$\iff$ $\mathcal{U} = \mathbb{V}(A)$ for some countable s.i. algebra A
$\iff$ $\mathcal{V} = \mathrm{Mod}\{\varepsilon\}$ for some equation

A is called a **splitting algebra** and
ε the **conjugate equation** of the **conjugate variety** $\mathcal{V}$.

$(\mathbb{V}(H), \mathcal{OML})$ is a splitting pair since an ortholattice has H as a sublattice $\iff$ it is not orthomodular.

The lattice $\Lambda_{\mathcal{OL}}$ of varieties of ortholattices

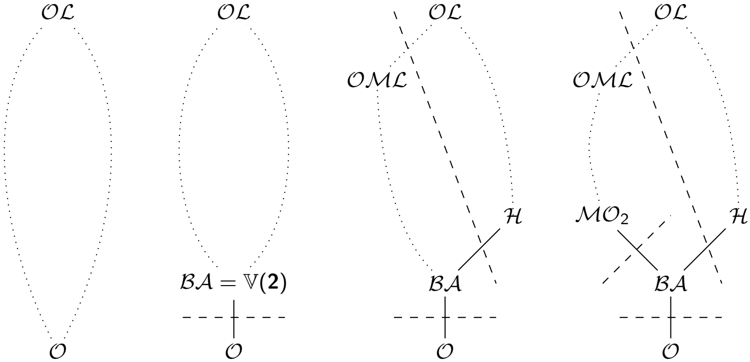


Figure: Some splittings in the lattice $\Lambda_{\mathcal{OL}}$ of ortholattice varieties

Generating varieties of ortholattices

$\mathbb{V}(A) = \mathbb{HSP}(A)$ is the **smallest variety** containing A

Examples: $\mathcal{MO}_n = \mathbb{V}(MO_n)$

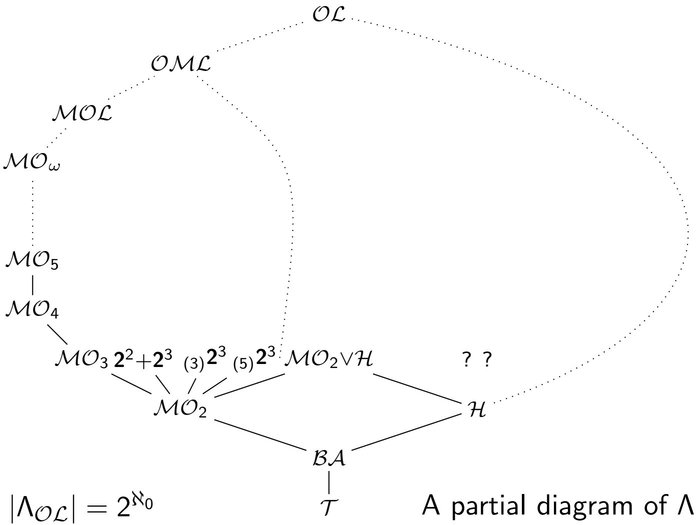
$\mathcal{H} = \mathbb{V}(H)$ the variety generated by the **hexagon benzene ring**.

$\Lambda_{\mathcal{OL}} =$ the **complete lattice** of all ortholattice varieties

$L + M =$ the **horizontal sum** of (ortho)lattices L, M

$_{(k)}\mathbf{2}^n = k$ glued copies of the finite **BA with n atoms**

The lattice $\Lambda_{\mathcal{OL}}$ of varieties of ortholattices



$|\Lambda_{\mathcal{OL}}| = 2^{\aleph_0}$

A partial diagram of $\Lambda_{\mathcal{OL}}$

Equational bases for some varieties

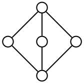
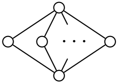
Baker [1972] proved that any **congruence distributive variety** that is generated by a finite algebra has a **finite equational basis**.

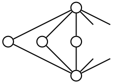
For bounded lattices L, M the (glued) **horizontal sum** $L +_h M$ is the disjoint union with the bounds identified. If L, M are ortholattices, so is $L +_h M$, and the **orthomodular identity is preserved**.

Bruns and Kalmbach [1971] found equational bases for all varieties of orthomodular lattices that are generated by **finite horizontal sums of finite Boolean algebras**.

In particular, $\mathcal{MO}_2$ has a **3-variable equational basis** $c(x,y)+c(x,z)+c(y,z)=1$, where $c(x,y) = xy+x'y+xy'+x'y'$.

Lattice equational bases for M_n, MO_n

$M_3 =$  $M_n =$ 

$M_\omega =$ 

Jónsson [1968] $\mathcal{M}_\omega$ has basis $E = \{w(x+yz)(y+z) \leqslant x+wy+wz\}$

$\mathcal{MO}_\omega$ has the same **lattice basis** relative to $\mathcal{OL}$.

$\mathcal{M}_n$ has basis $E_n = E \cup \{w \cdot \prod_{1\leqslant i<j\leqslant n}(x_i + x_j) \leqslant wx_1 + wx_2 + \cdots + wx_n\}$

E.g. $\mathcal{M}_3$ has basis $w(x_1 + x_2)(x_1 + x_3)(x_2 + x_3) \leqslant wx_1 + wx_2 + wx_3$

$\mathcal{M}_4$ has a **5-variable basis**, and $\mathcal{MO}_2$ has the **same lattice** basis.

$\mathcal{MO}_n$ has a $2n+1$-**variable lattice basis** E_{2n}.

An equational basis for the hexagon variety $\mathcal{H}$?

In Sept 2020 **John Harding** sent me an email about finding an equational basis for $\mathcal{H}$.

Kirby Baker's finite basis theorem is in principle **constructive**, but in practice not feasible even for very small algebras.

Roberto Giuntini proposed a 3-variable basis

$$B = \{(x+y)(x+z)(x'+yz) = (x+yz)(x'+yz),$$
$$(x+y)(x'+y)+xy' = x+y\}$$

McKenzie [1972] found a 4-variable basis for the lattice variety $\mathcal{N}_5$

$$M = \{w(x+y)(x+z) \leqslant w(x+yz)+wy+wz,$$
$$w(x+y(w+z)) = w(x+wy)+w(wx+yz)\}$$

We also investigated whether this is a basis for $\mathcal{H}$, but (at that time) no progress after a few weeks.

When is an OL variety defined by lattice equations?

Joint work with **J.B. Nation and Ralph Freese** (Jan 2022).

RdK denotes the **lattice reduct** of an ortholattice K.

Let $\Lambda_{\mathcal{L}}$ be the lattice of varieties of lattices and define $\rho : \Lambda_{\mathcal{OL}} \to \Lambda_{\mathcal{L}}$ by $\rho(\mathcal{V}) = \mathbb{V}(\{\mathrm{Rd}K \mid K \in \mathcal{V}\})$.

(i) Describe the range of ρ.

(ii) When is a variety $\mathcal{V}$ of ortholattices determined by an equational basis of $\rho(\mathcal{V})$?

Note: Varieties in the range of ρ are **self-dual**.

If k is odd then $\mathbb{V}(M_k)$ is **not** in the range of ρ.

An embedding $h : L \hookrightarrow \prod L_i$ is **subdirect** if $(\pi_i \circ h)[L] = L_i$ for all $i \in I$

L is **subdirectly irreducible** if $L \overset{sd}{\hookrightarrow} \prod L_i$ implies $L \cong L_i$ for some $i \in I$

Theorem

Let L be a finite s.i. lattice. Then L is a lattice-subdirect factor of an ortholattice if and only if there exists an ortholattice S such that $\mathrm{Rd}S \overset{sd}{\hookrightarrow} L \times L^d$, where L^d is the dual of L.

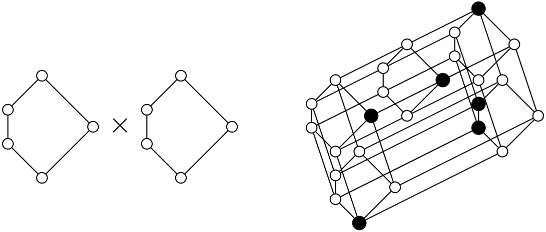
Proof (outline).

Let $K \in \mathcal{OL}$ and θ a lattice congruence with $(\mathrm{Rd}K)/\theta \cong L$.
On K define θ' by $x\theta' y \iff x'\theta y'$.
Then θ' is a lattice congruence (by De Morgan's law),
$(\mathrm{Rd}K)/\theta' \cong L^d$ and $\theta \cap \theta'$ is an ortholattice congruence
(since $x\theta\cap\theta' y \iff x'\theta'\cap\theta y'$). So take $S = K/\theta\cap\theta'$, then
$\mathrm{Rd}S \overset{sd}{\hookrightarrow} \mathrm{Rd}K/\theta \times \mathrm{Rd}K/\theta' \cong L \times L^d$. □

Deciding if $\mathbb{V}(L \times L^d)$ is in the range of ρ

For a finite s.i. lattice L, check if there exists a **subdirectly** embedded sublattice S of $L \times L^d$ that supports an orthocomplement.

Example: $\mathbb{V}(N_5 \times N_5^d) = \mathbb{V}(N_5) = \rho(\mathbb{V}(H))$ since $H \overset{sd}{\hookrightarrow} N_5 \times N_5^d$.



Any lattice basis for $\mathbb{V}(N_5)$ is a basis for $\mathbb{V}(H)$

Let K be an ortholattice such that $\mathrm{Rd}K \in \mathbb{V}(N_5)$.

Then $\mathrm{Rd}K$ has a subdirect embedding into a product of copies of N_5 and 2.

As in the proof of the preceding theorem, every N_5-congruence $\theta \in \mathrm{Con}(\mathrm{Rd}K)$ is paired with $\theta' = \{(x, y) \mid x'\theta y'\}$, and $\bar{\theta} := \theta \cap \theta'$ is an ortholattice congruence.

Thus we get an embedding of K into a product of $K/\bar{\theta}$ and copies of 2, where θ ranges over all N_5-congruences.

Since $K/\bar{\theta}$ is an orthocomplemented sublattice of $N_5 \times N_5$, it suffices to check that all subdirect sublattices of $N_5 \times N_5$ that admit an orthocomplement are isomorphic to H.

Any lattice basis for $\mathbb{V}(N(L))$ is a basis for $\mathbb{V}(L + L^d)$

This was first checked with a computer calculation for $N_5 \times N_5$.

Later generalized by hand to cover all lattices $N(L) = L +_p \{c\}$ where L is a finite subdirectly irreducible lattice.

(For lattices L, M the (loose) **parallel sum** $L +_p M$ is the disjoint union of L and M with a **new** $0, 1$ added.)

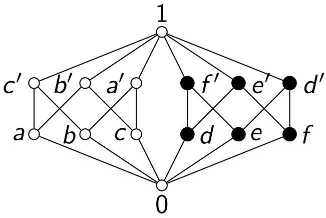
Note: $L +_p L^d$ is orthocomplemented by the map $x \leftrightarrow x^d$, $0 \leftrightarrow 1$.

Theorem

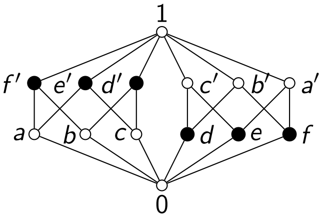
For any finite subdirectly irreducible lattice L, the ortholattice variety $\mathbb{V}(L +_p L^d)$ is determined by lattice identities.

$\mathcal{H}$ is covered by the case when $L = 2$.

Lattices with several (nonisomorphic) orthocomplements



$2^3 + 2^3$

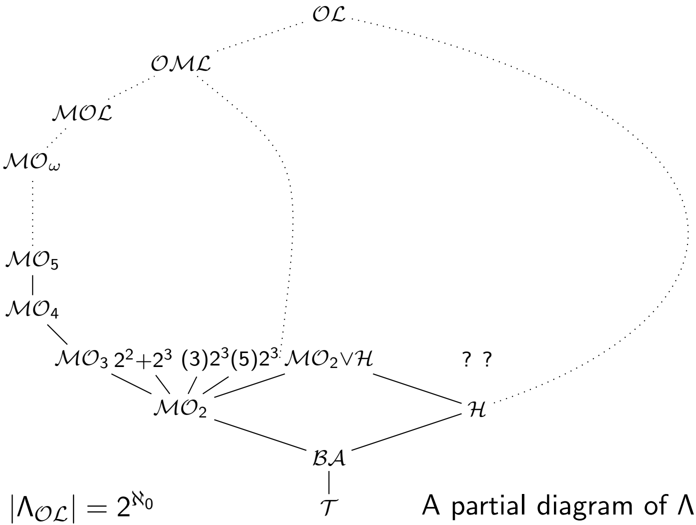


K

These two ortholattices cannot be distinguished by lattice identities.

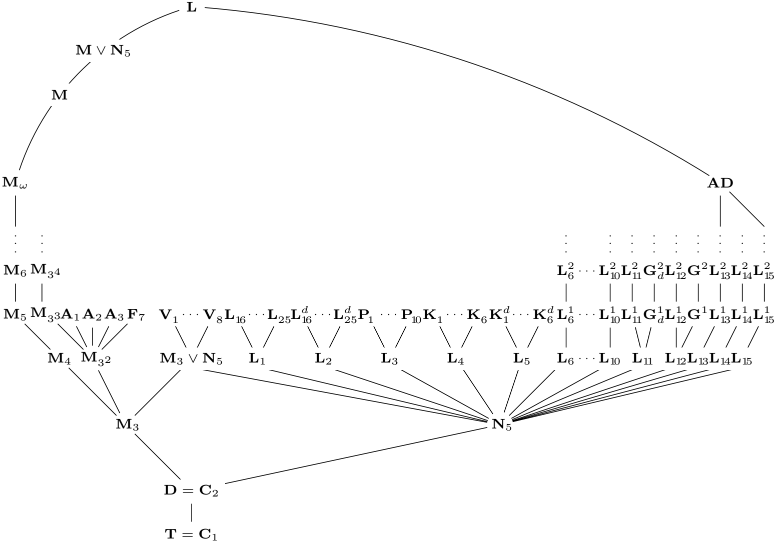
However $2^3 + 2^3$ is orthomodular, whereas H is a subalgebra of K.

Recall: $\Lambda_{\mathcal{OL}}$ lattice of ortholattice varieties



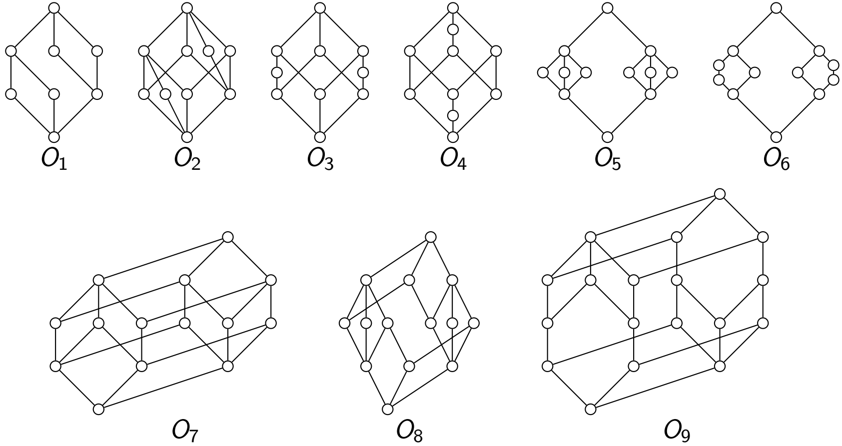
$|\Lambda_{\mathcal{OL}}| = 2^{\aleph_0}$

A partial diagram of $\Lambda_{\mathcal{OL}}$



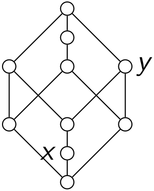
Compare with the lattice $\Lambda_{\mathbf{L}}$ of lattice varieties

Nine ortholattices that generate covers of $\mathbb{V}(H)$



$\mathcal{O}_5$ shows that a basis for $\mathbb{V}(H)$ requires 3 variables.

O_4 is a splitting ortholattice



O_4

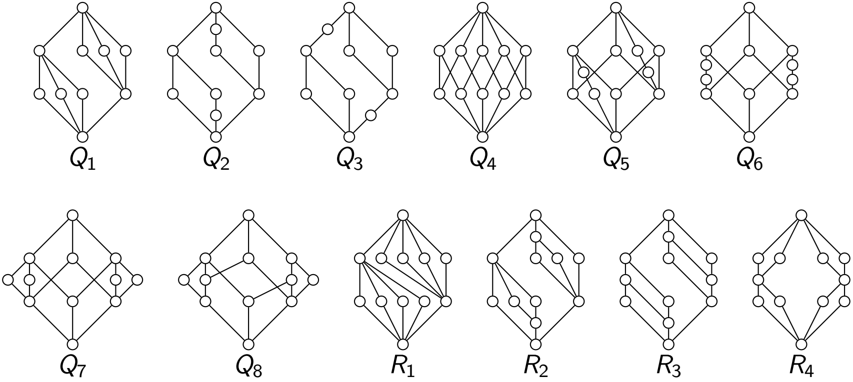
Theorem

For a variety $\mathcal{V}$ of ortholattices,
$O_4 \notin \mathcal{V} \iff \mathcal{V}$ *satisfies* $(x + x'y')(x + x'(x + y)) = x$.

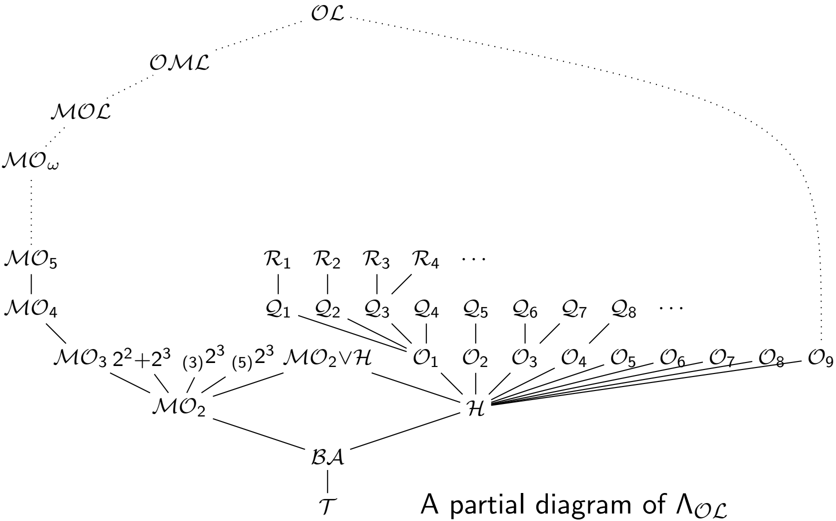
Equivalently, $\mathcal{V}$ satisfies $x \leqslant y \implies (x + y')(x + x'y) = x$.

This result was first proved with the help of Prover9.

Other subdirectly irreducible ortholattices



More details of the lattice $\Lambda_{\mathcal{OL}}$ of ortholattice varieties



A partial diagram of $\Lambda_{\mathcal{OL}}$

A full list of covering varieties gives a test for bases

Suppose $\mathcal{V}$ is a variety and $\mathcal{C}$ is a collection of varieties that **strongly cover** $\mathcal{V}$, i.e. for all varieties $\mathcal{W}$, $\mathcal{V} \subseteq \mathcal{W}$ implies $\mathcal{U} \subseteq \mathcal{W}$ for some $\mathcal{U} \in \mathcal{C}$.

Then E is a basis for $\mathcal{V}$ **iff** $\mathcal{V} \models E$ and for all $\mathcal{U} \in \mathcal{C}$, $\mathcal{U} \not\models E$.

Jónsson and Rival [1979] $\mathcal{M}_3 \vee \mathcal{N}_5, \mathbb{V}(L_1), \ldots, \mathbb{V}(L_{15})$ strongly cover $\mathcal{N}_5$. ($L_1, \ldots, L_{15}$ were found by **McKenzie** [1972].)

$\Rightarrow$ can easily test lattice identities to see if they are a basis for $\mathcal{N}_5$.

If so, then by the **preceding results** they are also a basis for $\mathcal{H}$.

But to test ortholattice identities we need a full list of covers of $\mathcal{H}$

Is $\mathcal{MO}_2 \vee \mathcal{H}, \mathcal{O}_1, ..., \mathcal{O}_9$ a full list of covers of $\mathcal{H}$?

So far we have proved the following result.

Theorem

If a finite ortholattice K has an atom a such that $\downarrow a'$ is not a prime ideal, then there exists $x \in K$ such that $Sg(a, x)$ contains MO_2 or O_j for some $j \in \{1, 2, 3, 4, 8\}$.

Now can assume that K is a finite ortholattice in which $\downarrow a'$ is a prime ideal for every atom a. If $K \notin \mathcal{H}$ then show K contains MO_2 or O_j for some $j \in \{4, 5, 6, 7, 9\}$.

Last step would be to remove finiteness of K.

If $\mathcal{MO}_2 \vee \mathcal{H}, \mathcal{O}_1, ..., \mathcal{O}_9$ is a full list of covers of $\mathcal{H}$ then **Roberto Giuntini's identities B are also a basis for $\mathcal{H}$.**

Some references for Part 1

- K. Baker, Finite equational bases for finite algebras in a congruence-distributive equational class, Advances in Math. 24 (1977), 207–243.
- G. Bruns and G. Kalmbach: Varieties of orthomodular lattices, Canadian J. Math., Vol. XXIII, No. 5, 1971, pp. 802–810
- B. Jónsson: Equational classes of lattices. Math. Scand., 22:187–196, 1968.
- B. Jónsson and I. Rival: Lattice varieties covering the smallest nonmodular variety. Pacific J. Math., 82(2):463–478, 1979.
- R. McKenzie: Equational bases and nonmodular lattice varieties. Trans. Amer. Math. Soc., 174:1–43, 1972.

Part 2

Joint work with **Melissa Sugimoto** at CUNY,

José Gil-Férez and **Sid Lodhia** at Chapman

Plonka sums of integral involutive partially ordered monoids

Involutive residuated lattices and posets

A **pointed residuated lattice** $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /, 0)$ is a lattice $(A, \wedge, \vee)$ and a monoid $(A, \cdot, 1)$ with a constant 0 such that

$$xy \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z.$$

It is **involutive** if $-{\sim}x = x = {\sim}{-x}$ where $-x = 0/x$, ${\sim}x = x\backslash 0$.

In this case $x/y = -(y \cdot {\sim}x)$ and $x\backslash y = {\sim}(-y \cdot x)$, so the residuals become term-definable.

Examples: **Boolean algebras** $(xy = x \wedge y)$ and **MV-algebras**

$\mathcal{MV} = \mathbb{V}\{([0,1], \min, \max, \odot, 1, \sim)\}$ where $[0,1] \subseteq \mathbb{R}$,
$x \odot y = \max(x + y - 1, 0)$ and $-x = {\sim}x = 1 - x$

Involutive residuated posets = **ipo-monoids** generalize involutive residuated lattices by replacing $(A, \wedge, \vee)$ with $(A, \leqslant)$

Structural results about involutive residuated lattices?

The structure of (finite) Boolean algebras is well understood.

Similarly for MV-algebras, they are products of MV-chains $\mathbf{MV}_n$

Can we build on these results to describe larger classes of involutive residuated lattices?

Boolean algebras are idempotent $xx = x$, so study these involutive RLs.

Full description for the finite commutative ones by [J., Tuyt & Valota 2020]

Used Mace4 to look at finite models up to size 16.

Idempotent ipo-monoids are Plonka sums of BAs

Let $\{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j \colon i \leqslant j\}$ be a family of homomorphisms indexed by a join-semilattice $(I, \vee, \bot)$ and **compatible**, i.e., $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, if $i \leqslant j \leqslant k$, and φ_{ii} is the identity on $\mathbf{A}_i$.

Its **Płonka sum** is the algebra $\mathbf{S}$ with universe $\biguplus_{i \in I} A_i$ and

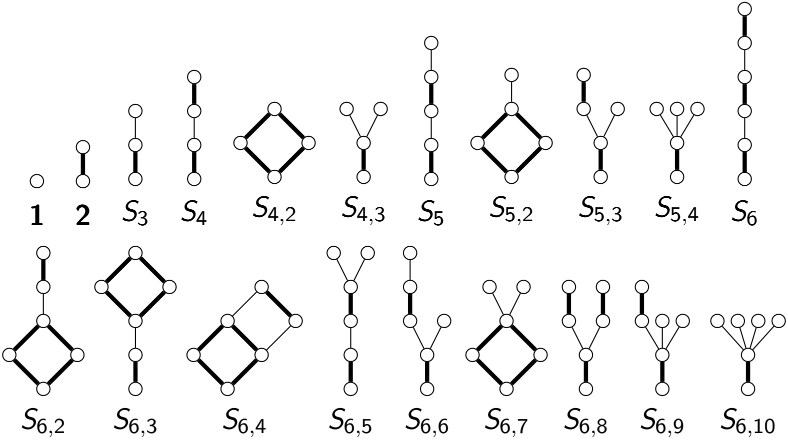
$a \cdot^{\mathbf{S}} b = \varphi_{ik}(a) \cdot^{\mathbf{A}_k} \varphi_{jk}(b)$ where $a \in A_i, b \in A_j, k = i \vee j$

$\sim^{\mathbf{S}} a = \sim^{\mathbf{A}_i} a$, $-^{\mathbf{S}} a = -^{\mathbf{A}_i} a$ and $1^{\mathbf{S}} = 1^{\mathbf{A}_\bot}$.

Lemma (J., Sugimoto)

The $\cdot$ of every finite commutative idempotent ipo-monoid $\mathbf{A}$ is a Płonka sum of generalized BA homomorphisms $\varphi_{pq}(x) = xq$ indexed by $I = \{p \in A \colon p \geq 1\}$ where $\mathbf{A}_p = \{x \in A \colon x/x = p\}$.

Mace4 produced the following diagrams for the **monoidal order** $x \sqsubseteq y \iff x \cdot y = x$ of all **commutative idempotent ipo-semigroups** (bold lines show Boolean components)



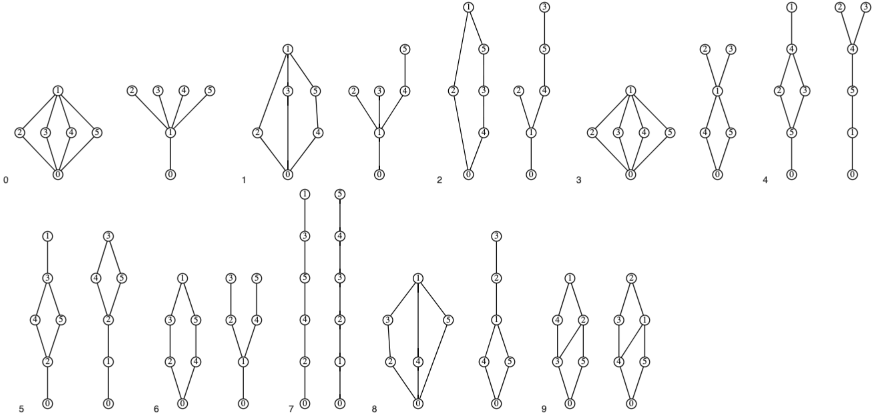
How does this look in Prover9 using Colab?

Go to http://colab.research.google.com and paste:

```
!pip install provers
!git clone https://github.com/jipsen/Prover9.git
from provers import *; execfile("/content/Prover9/Prover9.py")

iposg=[
"x<=x", "x<=y & y<=x -> x=y", "x<=y & y<=z -> x<=z",
"x*y <= z <-> x<=-(y*~z)", "x*y <= z <-> y<=~(-z*x)",
"x<=y -> -y<=-x", "x<=y -> ~y<=~x", "-~x = x", "~-x = x",
"(x*y)*z = x*(y*z)"]
b=p9(iposg+["x*x=x","0*x=0"],[],100,100,[9])
show(b[6])
```

Commutative idempotent ipo-semigroups of size 6



However the previous result did **not** explain how to reconstruct the partial order $\leqslant$ of the ipo-monoid $\mathbf{A}$ and did **not** characterize the families of homomorphisms.

With Sid Lodhia we investigated weaker axioms than assuming commutativity and idempotence.

The components $\mathbf{A}_p$ have top element $1_p = p = -(x \cdot {\sim}x)$, hence they are **integral** and have bottom $0_p = x \cdot {\sim}x$.

Prover9 was helpful in showing the following axioms suffice:

An ipo-monoid is **locally integral** if it satisfies

(i) $x \cdot {\sim}x = -x \cdot x$, (ii) $xx \leqslant x$ and (iii) $x \leqslant 0 \Rightarrow xx = x$

Every integral (i.e., $x \leqslant 1$) ipo-monoid is locally integral.

Locally integral ipo-monoids

An **ipo-monoid** $(A, \leqslant, \cdot, 1, \sim, -)$ is a poset $(A, \leqslant)$ and a monoid $(A, \cdot, 1)$ with $0 = {\sim}1 = -1$ such that

$$x \leqslant y \iff x \cdot {\sim}y \leqslant 0 \iff -y \cdot x \leqslant 0$$

It follows that ${\sim}{-}x = x = -{\sim}x$ and $x \leqslant {\sim}y \iff y \leqslant -x$.

The class of ipo-monoids includes **all groups** (if $\leqslant$ is $=$) and

all partially ordered groups where ${\sim}x = -x = x^{-1}$, $0 = 1$.

Boolean algebras and **MV-algebras** are integral ipo-monoids, in fact **iℓ-monoids** ($\vee, \wedge$ are definable)

Structural Characterization of Locally Integral ipo-monoids

Theorem (Gil-Férez, J., Lodhia)

Let $\mathbf{A}$ *be a locally integral ipo-monoid and* $\{\varphi_{pq} : p \leqslant q\}$ *as before.*

Then their Płonka sum $\left(\biguplus A_p, \cdot^{\mathbf{S}}, 1^{\mathbf{S}}\right)$ *is the monoidal reduct of* $\mathbf{A}$*.*

Define ${\sim}^{\mathbf{S}}x = {\sim}^{\mathbf{A}_p}x$ *and* $-^{\mathbf{S}}x = -^{\mathbf{A}_p}x$*, for every* $x \in A_p$*.*

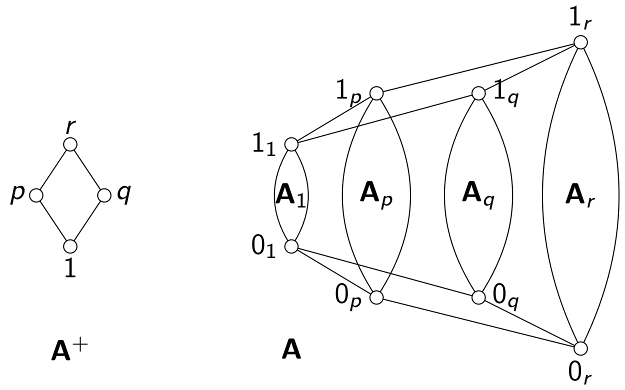
Define $x \leqslant^{\mathbf{S}} y \iff x \cdot^{\mathbf{S}} {\sim}^{\mathbf{S}}y = 0_{pq}, \quad$ *for all* $x \in A_p$, $y \in A_q$*.*

Then $\left(\biguplus A_p, \leqslant^{\mathbf{S}}, \cdot^{\mathbf{S}}, {\sim}^{\mathbf{S}}, -^{\mathbf{S}}\right) = \mathbf{A}$*.*

Moreover, if $\mathbf{A}$ *is in InRL then all* $\mathbf{A}_p$ *are in InRL.*

Furthermore, $\mathbf{A}$ *is commutative if and only if all its components are commutative*

A Generic Example with 4 Integral InRL Components



Glueing Integral ipo-monoids

Let $(D, \vee, 1)$ be a lower-bounded join-semilattice;
$\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \sim_p, -_p)$ integral ipo-monoid, for every $p \in D$;
$\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^D q\}$ compat. family of monoidal hom.

Define the structure:

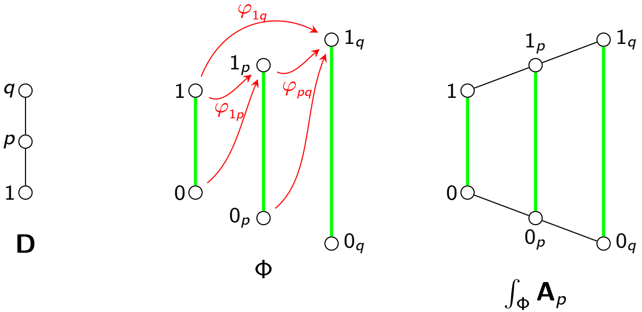
$$\int_\Phi \mathbf{A}_p = \left(\biguplus_D A_p, \leqslant^G, \cdot^G, 1^G, \sim^G, -^G \right)$$

where $\left(\biguplus_D A_p, \cdot^G, 1^G \right)$ is the Płonka sum of the family Φ
and for all $p, q \in D$, $a \in A_p$, and $b \in A_q$,

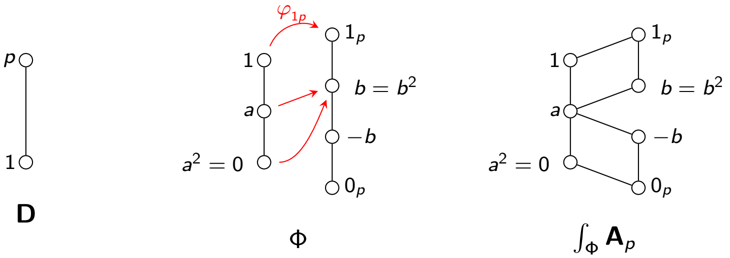
- $\sim^G a = \sim_p a$ and $-^G a = -_p a$,
- $a \leqslant^G b \iff a \cdot^G \sim^G b = 0_{p \vee q}$.

$\int_\Phi \mathbf{A}_p$ is the **glueing of** $\{\mathbf{A}_p : p \in D\}$ **along the family** Φ.

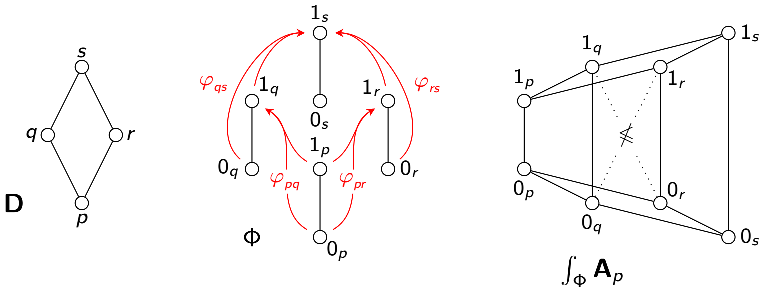
A Sugihara Glueing of Copies of the Standard MV-chain



Glueing Ł$_3$ into a Small IMTL-algebra



Glueing of Integral ipo-monoids that is not an ipo-monoid



The relation $\leqslant$ of $\int_\Phi \mathbf{A}_p$ is not transitive.

Required Conditions for Glueing Integral ipo-monoids

(balanced): for all $p, q \in D, a \in A_p, b \in A_q$,

$$a \cdot^G \sim^G b = 0_{p \vee q} \iff -^G b \cdot^G a = 0_{p \vee q}.$$

(zero): for all $p \leqslant^D q$, $\quad \varphi_{pq}(0_p) = 0_q \iff p = q$.

(tr): for all $a, b, c \in \biguplus A_p$, $\quad$ if $a \leqslant^G b$ and $b \leqslant^G c$, then $a \leqslant^G c$.

Main Glueing Result

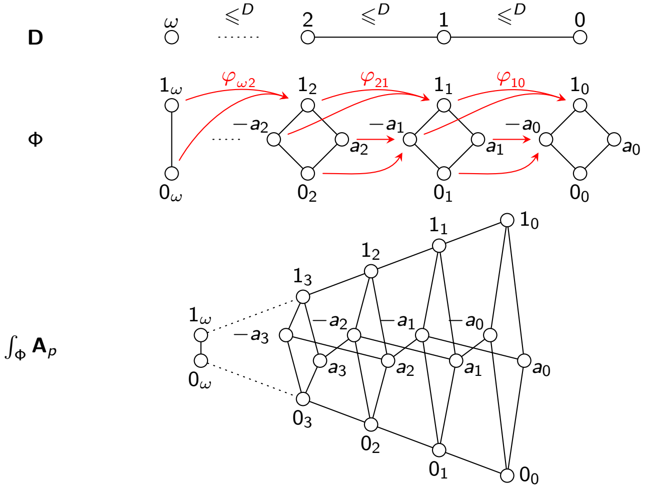
Theorem (Gil-Férez, J., Lodhia)

A structure $\mathbf{A}$ *is a locally integral ipo-monoid if and only if there is*

- *a lower-bounded join-semilattice* $\mathbf{D}$,
- *a family of integral ipo-monoids* $\{\mathbf{A}_p : p \in D\}$, *and*
- *a compatible family* $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^{\mathbf{D}} q\}$ *of monoidal homomorphisms satisfying (bal), (zero), and (tr)*

so that $\mathbf{A} = \int_\Phi \mathbf{A}_p$.

Glueing of infinitely many BAs that produces an $i\ell$-monoid



A Few Remarks and Questions

The condition (tr) can be replaced by more "local" condition.

- for all $p \leqslant^{\mathbf{D}} q$, and $a, b \in A_p$, $a \leqslant_p b \implies \varphi_{pq}(a) \leqslant_q \varphi_{pq}(b)$; (mon)
- for all $p \leqslant^{\mathbf{D}} q$, $p \leqslant^{\mathbf{D}} r$, and $a \in A_p$, ${\sim}\varphi_{pq}(a) \leqslant^{\mathbf{G}} \varphi_{pr}({\sim}a)$; (lax)
- for all $p \vee r \leqslant^{\mathbf{D}} v$, $a \in A_p$, and $b \in A_r$,

$$\varphi_{rv}({\sim}b) \leqslant_v {\sim}\varphi_{pv}(a) \implies a \leqslant^{\mathbf{G}} b. \qquad ({\sim}\text{lax})$$

A locally integral ipo-monoid **A** is idempotent if and only if all its integral components are Boolean algebras.

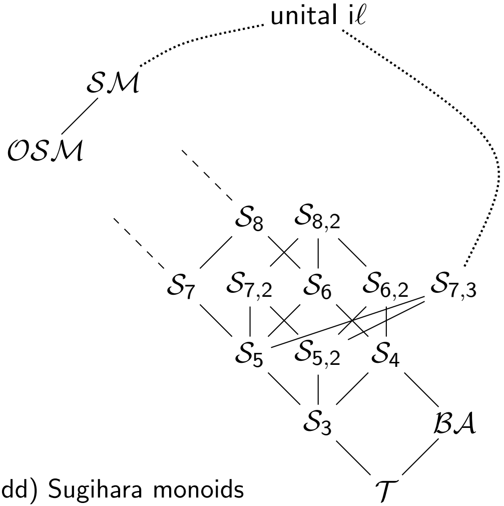
Several properties are "local" (i.e., **A** satisfies them if and only if all its components do): e.g., commutativity, local finiteness.

Under which conditions is **A** lattice-ordered?

Are locally integral ipo-monoids or InRLs decidable?

Some subvarieties of commutative idempotent InRLs

Let $\mathcal{S}_{i,j} = \mathbb{V}(S_{i,j})$.



$(\mathcal{O})\mathcal{SM}$ = (odd) Sugihara monoids

Some equational bases for commutative idempotent InRLs

The previous diagram is complete below $\mathcal{SM}$ and $\mathcal{S}_{5,2}$.

Hence we have full lists of covering varieties for proper subvarieties of $\mathcal{SM}$ (excluding $\mathcal{OSM}$).

$\mathcal{BA}$ is covered only by $\mathcal{S}_3 \vee \mathcal{BA}$, so $x0 = 0$ is a basis relative to $\mathcal{SM}$

$\mathcal{S}_3$ has $(x \vee -x)(0 \vee -y) = x \vee -(xy)$ as basis relative to $\mathcal{OSM}$.

$\mathcal{S}_4$ has $0 \leqslant x \vee -(xy)$ as basis relative to $\mathcal{SM}$.

$\mathcal{S}_{5,2}$ has $(x \vee -x)(0 \vee -y) = x \vee -(xy)$ as basis relative to odd unital iℓ-semilattices.

Dual Representation by Partial Functions Between Sets

Partial Functions

Definition. A **proper partial function** $f : X \to Y$ is a function from U to Y where $U \subsetneq X$ is the domain of f.

Developing a Dual Representation

Given a commutative idempotent ipo-monoid **A**, it is Płonka sum of Boolean components.

Each Boolean component is determined by its set of atoms.

The partial functions map between sets of atoms (opposite to homomorphisms).

A dual representation of families of Boolean algebras gives a much more compact way of drawing finite ipo-monoids.

Dual Representation by Partial Functions Between Sets

Every finite Boolean algebra $\mathbf{A}_i$ is **isomorphic** to the powerset Boolean algebra of its finite set X_i of atoms.

For $i \leqslant j$, the **generalized BA homomorphism** h_{ji} corresponds to the **partial map** $f_{ij} : X_i \to X_j$ defined by

$$f_{ij}(a) = b \iff a \leqslant h_{ji}(b) \text{ and } a \nleqslant h_{ji}(0_j).$$

A **family of proper partial maps** is a triple $\mathbf{X} = (X_i, f_{ij}, I)$ st

- for a semilattice I, $\{X_i : i \in I\}$ is a family of disjoint sets, and
- $f_{ij} : X_i \to X_j$ is a proper partial map for all $i \leqslant j \in I$ such that $f_{ii} = id_{X_i}$ and for all $i \leqslant j \leqslant k$, $f_{jk} \circ f_{ij} = f_{ik}$.

$\Rightarrow$ Every commutative idempotent ipo-monoid can be represented by a family of proper partial maps.

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THANKS!