Splittings and a (yet another!) generalization of Baker's Finite Basis Theorem

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Overview

Part 1: some general theory of splittings in quasivarieties

Part 2: what to do with it.

Part 1: Splitting pairs

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From then on the concept has been applied many times to the lattice of subvarieties of a given variety; however there are almost no application to the lattice of subquasivarieties of a quasivariety.

Three definitions

Let Q be a quasivariety and $\mathbf{A} \in$ Q; a Q-**congruence** of $\mathbf{A}$ is a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta \in$ Q. The Q-congruences of $\mathbf{A}$ form an algebraic lattice denoted by $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$.

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Suppose that (Q_1, Q_2) is a splitting in $\Lambda_q(\mathsf{Q})$; if Σ_1 is the quasiequational theory of Q_1 , then

$$\mathsf{Q}_1 = \mathrm{Mod}(\Sigma_1) = \bigcap \{\mathrm{Mod}(\sigma) : \sigma \in \Sigma_1\}.$$

As Q_1 is completely meet prime there exists a σ_1 such that if $\Sigma \vdash \sigma_1$, then there is some $\sigma \in \Sigma$ such that $\sigma \vdash \sigma_1$; in particular $\mathsf{Q}_1 = \mathrm{Mod}(\sigma_1)$.

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It follows that

$$\mathsf{Q}_2 = \bigvee\{\mathbb{Q}(\mathbf{A}) : \mathbf{A} \text{ is Q-irreducible and finitely generated}\};$$

as Q_2 is completely join prime $\mathsf{Q}_2 = \mathbb{Q}(\mathbf{A})$ for some finitely generated Q-irreducible algebra $\mathbf{A} \in \mathsf{Q}$.

Q-splitting algebras

A Q-**splitting algebra** is a finitely generated Q-irreducible algebra $\mathbf{A} \in \mathsf{Q}$ such that there is a $\mathsf{Q}_{\mathbf{A}} \subseteq \mathsf{Q}$ such that $(\mathsf{Q}_{\mathbf{A}}, \mathbb{Q}(\mathbf{A}))$ is a splitting pair in $\Lambda_q(\mathsf{Q})$; in this case the single quasiequation axiomatizing $\mathsf{Q}_{\mathbf{A}}$ is called the **splitting quasiequation** for $\mathbf{A}$.

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In other words $\mathbf{A}$ is Q-splitting if there exists a largest subquasivariety $\mathsf{Q}_\mathbf{A}$ of Q, called the **conjugate quasivariety of A** such that $\mathbf{A} \notin \mathsf{Q}_\mathbf{A}$. As a consequence of what we have seen so far we have:

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Lemma

Let Q *be a quasivariety and* $(\mathsf{Q}_1, \mathsf{Q}_2)$ *be a splitting pair in* $\Lambda_q(\mathsf{Q})$*; then there exists a finitely generated* Q*-irreducible algebra* $\mathbf{A} \in \mathsf{Q}$ *such that* $(\mathsf{Q}_1, \mathsf{Q}_2) = (\mathsf{Q}_{\mathbf{A}}, \mathbb{Q}(\mathbf{A}))$.

The FEP

A quasivariety has the **finite extension property** (FEP) if for any algebra $\mathbf{A} \in \mathsf{Q}$ and for any finite partial subalgebra $\mathbf{F}$ of $\mathbf{A}$, there is a finite algebra $\mathbf{B} \in \mathsf{Q}$ with $\mathbf{F} \leq \mathbf{B}$.

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Lemma

If Q has the FEP, then every Q-splitting algebra is finite.

Q-finitely presented algebras

Let σ be a type and let K be a class of algebras of type σ; let X be a set of variables and Σ be a set of equations of type σ in variables from X. We say that the pair (X, Σ) is **a** K-**presentation of A** $\in$ K if there exists a function $\alpha : X \longrightarrow A$ such that

- $\alpha(X)$ generates **A** and for any $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n) \in \Sigma$,
$$p(\alpha(x_1), \dots, \alpha(x_n)) = q(\alpha(x_1), \dots, \alpha(x_n));$$
- if **B** $\in$ K and $\beta : X \longrightarrow B$ such that for any $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n) \in \Sigma$, $p(\beta(x_1), \dots, \beta(x_n)) = q(\beta(x_1), \dots, \beta(x_n))$, then there exists a homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ such that $f(\alpha(x)) = \beta(x)$ for all $x \in X$.

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An algebra **A** is Q-finitely presented if there is a Q-presentation (X, Σ) of **A** in which both X and Σ are finite.

Let (X, Σ) be a finite Q-presentation of Q-irreducible algebra **A** and let (p, q) any pair of elements generating the Q-monolith of **A**. Then the quasiidentity

$$\Sigma \Rightarrow p \approx q$$

is a **characteristic quasiidentity** of **A**. We denote it by $\mathrm{ch}(\mathbf{A})$ the set of characteristic quasiidentities of **A**.

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Lemma

If an algebra $\mathbf{A} \in \mathsf{Q}$ is Q*-irreducible and* Q*-finitely presented, then for every algebra $\mathbf{B} \in \mathsf{Q}$ and for every $\Phi \in \mathrm{ch}(\mathbf{A})$, $\mathbf{B} \not\models \Phi$ if and only if $\mathbf{A} \in \boldsymbol{IS}(\mathbf{B})$.*

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The lemma is the key of a classical argument from which (I swear!) we can prove that a Q-presentation is exactly what it should be,

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Theorem

1. *If* Q *is a quasivariety for any finite sets* X, Σ *there is exactly one algebra (up to isomorphism)* $\mathbf{A} \in \mathsf{Q}$ *that is* Q*-finitely presented by* (X, Σ).

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3. *If* $\vartheta_{\mathsf{Q}}(\Sigma)$ *is the* Q*-congruence generated by* Σ *in* $\mathbf{F}_{\mathsf{Q}}(X)$ *then*

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$$\mathbf{F}_{\mathsf{Q}}(X, \Sigma) \cong \mathbf{F}_{\mathsf{Q}}(X)/\vartheta_{\mathsf{Q}}(\Sigma).$$
4. *An algebra* $\mathbf{A} \in \mathsf{Q}$ *is* Q*-finitely presented if and only if* $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(X)/\vartheta_{\mathsf{Q}}(\Sigma)$ *for some finite sets* X, Σ.

Getting closer

Let Q be a quasivariety and $\mathbf{A} \in \mathsf{Q}$; we define

$$[\mathsf{Q} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{Q} : \mathbf{A} \notin \boldsymbol{IS}(\mathbf{B})\}.$$

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$$[\mathrm{Q} : \mathbf{A}] = \{\mathbf{B} \in \mathrm{Q} : \mathbf{A} \notin \boldsymbol{IS}(\mathbf{B})\}.$$

Lemma

Let Q be a quasivariety of finite type; if $\mathbf{A} \in \mathrm{Q}$ is Q-irreducible and Q-finitely presented, then $[\mathrm{Q} : \mathbf{A}]$ is a quasivariety. If $\mathbf{A}$ is also weakly projective in Q, then $[\mathrm{Q} : \mathbf{A}]$ is a variety. Moreover if Q is locally finite then both converse implications holds.

Proof.

Suppose that $\mathbf{A}$ is Q-irreducible and Q-finitely presented, i.e. $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(x)/\theta_{\mathsf{Q}}(\Sigma)$. If Φ is a characteristic quasiidentity of $\mathbf{A}$ then by the key Lemma we get at once that $[\mathsf{Q} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{Q} : \mathbf{B} \vDash \Phi\}$ and this of course implies that $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety. If $\mathbf{A}$ is also weakly projective and then if $\mathbf{B} \in \boldsymbol{H}(\mathbf{C})$ for some $\mathbf{C}$ in $[\mathsf{Q} : \mathbf{A}]$ and $\mathbf{A} \in \boldsymbol{IS}(B)$, then $\mathbf{A} \in \boldsymbol{SH}(\mathbf{C}) \subseteq \boldsymbol{HS}(\mathbf{C})$. Hence $\mathbf{A} \in \boldsymbol{IS}(\mathbf{C})$ which is impossible. So $\mathbf{A} \notin \boldsymbol{IS}(\mathbf{B})$ and $[\mathsf{Q} : \mathbf{A}]$ is a variety.

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Conversely suppose that Q is locally finite and $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety; then $\mathbf{A} \in \mathsf{Q}$ is embeddable in an ultraproduct of its finite subalgebras, say $\mathbf{A} \in \boldsymbol{ISP}_u(\{\mathbf{B}_i : i \in I\})$. If $\mathbf{A}$ is not finite, then $\mathbf{A} \notin \boldsymbol{IS}(\mathbf{B}_i)$ for all i, so $\mathbf{B}_i \in [\mathsf{Q} : \mathbf{A}]$ for all i. But then $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$, a contradiction. Hence $\mathbf{A}$ is finite and, begin Q of finite type, also Q-finitely presented.
Suppose now that $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{B}_i$ where each $\mathbf{B}_i$ is Q-irreducible in Q. Since $\mathbf{A}$ is finite, each $\mathbf{B}_i$ can be taken to be finite; if $\mathbf{A} \notin \boldsymbol{IS}(\mathbf{B}_i)$ for all i, then $\mathbf{B}_i \in [\mathsf{Q} : \mathbf{A}]$ for all i and as above we get a contradiction.

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Hence there is an i such that $\mathbf{A} \in \boldsymbol{IS}(\mathbf{B}_i)$, so that $|A| \leq |B_i|$; on the other hand $\mathbf{B}_i \in \boldsymbol{H}(\mathbf{A})$, so $|B_i| \leq |A|$. Since everything is finite we have $\mathbf{A} = \mathbf{B}_i$ and $\mathbf{A}$ is Q-irreducible.

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Suppose now that $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{B}_i$ where each $\mathbf{B}_i$ is Q-irreducible in Q. Since $\mathbf{A}$ is finite, each $\mathbf{B}_i$ can be taken to be finite; if $\mathbf{A} \notin \boldsymbol{IS}(\mathbf{B}_i)$ for all i, then $\mathbf{B}_i \in [\mathsf{Q} : \mathbf{A}]$ for all i and as above we get a contradiction.

Hence there is an i such that $\mathbf{A} \in \boldsymbol{IS}(\mathbf{B}_i)$, so that $|A| \leq |B_i|$; on the other hand $\mathbf{B}_i \in \boldsymbol{H}(\mathbf{A})$, so $|B_i| \leq |A|$. Since everything is finite we have $\mathbf{A} = \mathbf{B}_i$ and $\mathbf{A}$ is Q-irreducible.

Finally if $[\mathsf{Q} : \mathbf{A}]$ is a variety, let $\mathbf{B} \in \mathsf{Q}$ such that $\mathbf{A} \in \boldsymbol{H}(\mathbf{B})$; if $\mathbf{A} \notin \boldsymbol{IS}(\mathbf{B})$, then $\mathbf{B} \in [\mathsf{Q} : \mathbf{A}]$ and thus $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$ a contradiction. □

Too many splittings

Corollary

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Proof.

By the previous Lemma, $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety and $\mathbf{A} \notin [\mathsf{Q} : \mathbf{A}]$. Suppose that $\mathbf{A} \in \mathsf{Q}'$ and $\mathsf{Q}' \not\subseteq [\mathsf{Q} : \mathbf{A}]$. Then there is a $\mathbf{B} \in \mathsf{Q}'$ s.t. $\mathbf{A} \in \boldsymbol{IS}(\mathbf{B})$; but then $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$ a contradiction. Hence $\mathsf{Q}' \subseteq [\mathsf{Q} : \mathbf{A}]$ and $\mathbf{A}$ is splitting with $[\mathsf{Q} : \mathbf{A}]$ as conjugate variety. □

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However:

Lemma

Let Q be quasivariety; then if $\mathbf{A} \in \mathsf{Q}$ is finitely generated, weakly Q-projective and Q-irreducible, $\mathbf{A}$ is Q-splitting with conjugate variety $[\mathsf{Q} : \mathbf{A}]$.

Part 2: Ordering the splitting algebras

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Hence we may safely assume that all the algebras in Q_{spl} are Q-irreducible and that Q_{spl} is partially ordered by $\preceq$.

Two easy Lemmas

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Let Q be a quasivariety and let $S \subseteq \mathsf{Q}_{spl}$ be an antichain w.r.t. $\preceq$. If $S_1, S_2 \subseteq S$ and $S_1 \neq S_2$ then $\mathbb{Q}(S_1) \neq \mathbb{Q}(S_2)$.

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If Q is a quasivariety such that Q_{spl} contains an infinite antichain, then $\Lambda_q(\mathsf{Q})$ is not countable.

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If Q is a quasivariety such that Q_{spl} contains an infinite antichain, then $\Lambda_q(\mathsf{Q})$ is not countable.

Lemma

Let Q be a quasivariety and let $A \subseteq \mathsf{Q}_{spl}$ a class of finite algebras; then A contains a minimal antichain.

Equational bases

Let Q be a quasivariety; an **equational basis relative to** Q of a subquasivariety $Q' \subseteq Q$ is a set Σ of equations such that $Q' = \mathrm{Mod}(\Sigma) \cap Q$.

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The following is essentially due to V. Gorbunov:

Theorem

(Gorbunov) If Q is primitive, then any Q-finitely presented and Q-irreducible algebra in Q is weakly projective in Q.

Primitivity implies splitting equations

Observe that if a quasivariety is primitive and **A** is Q-splitting, then its conjugate quasivariety $\mathsf{Q}_{\mathbf{A}}$ can be defined relative to Q by a set Δ of equations; therefore if Σ is a quasiquational basis for Σ

$$\mathsf{Q}_{\mathbf{A}} = \operatorname{Mod}(\Sigma \cup \Delta) = \bigcap_{\delta \in \Delta} \operatorname{Mod}(\Sigma \cup \{\delta\}).$$

But $\mathsf{Q}_{\mathbf{A}}$ is strictly meet prime, so there is a $\delta \in \Delta$ such that $\mathsf{Q}_{\mathbf{A}} = \operatorname{Mod}(\Sigma \cup \{\delta\})$.

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Observe also that if **A** is Q-irreducible and Q-finitely presented then **A** is splitting and also, by Gorbunov's result, weakly projective. Thus $\mathsf{Q}_{\mathbf{A}} = [\mathsf{Q} : \mathbf{A}]$ and the latter is indeed a variety by the key Lemma.

The main result

If $\mathsf{Q}' \subseteq \mathsf{Q}$, then $I[\mathsf{Q}', \mathsf{Q}] = \{\mathsf{Q}'' : \mathsf{Q}' \subseteq \mathsf{Q}'' \subseteq \mathsf{Q}\}$.

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Theorem

Let Q be a primitive quasivariety of finite type and $\mathsf{Q}' \subseteq \mathsf{Q}$ such that every quasivariety in $I[\mathsf{Q}', \mathsf{Q}]$ has the FEP. Then the following are equivalent:

1. *every $\mathsf{Q}'' \in I[\mathsf{Q}', \mathsf{Q}]$ has a finite equational basis relative to Q;*
2. *$I[\mathsf{Q}', \mathsf{Q}]$ is countable;*
3. *$\mathsf{Q}_{spl} \setminus \mathsf{Q}'_{spl}$ has no infinite antichain;*
4. *$I[\mathsf{Q}', \mathsf{Q}]$ enjoys the descending chain condition.*

The Corollary

If Q is locally finite, then all the subquasivarieties of Q have the FEP, hence

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For a primitive locally finite quasivariety Q of finite type the following are equivalent:

1. *every subquasivariety of Q has a finite equational basis relative to Q;*
2. $\Lambda_q(\mathsf{Q})$ *is countable;*
3. Q_{spl} *has no infinite antichain;*
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Corollary

Let Q *be a primitive quasivariety of finite type; if* Q *has finitely many* Q*-irreducible algebras, then every subquasivariety of* Q *has a finite equational basis relative to* Q*.*

Rybakov's example

The Corollary can be seen as a version of Baker's Finite Basis Theorem for quasivarieties; for instance our version differs from the one in Pigozzi (1988) in that we drop Q-congruence distributivity and add primitivity, which in turn gives us an equational basis.

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Note that primitivity is essential; Rybakov (1997) produced a finite Heyting algebra $\mathbf{A}$ that does not have a finite *quasiequational* basis relative to the variety H of Heyting algebras. As $\boldsymbol{V}(\mathbf{A})$ is congruence distributive, Baker's Finite Basis Theorem applies, and $\boldsymbol{V}(\mathbf{A})$ has a finite equational basis relative to H. It follows that $\mathbb{Q}(\mathbf{A})$ cannot have a finite quasiequational basis relative to $\boldsymbol{V}(\mathbf{A})$.

THANK YOU!