(Congruence) diameter of semigroups

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Prelude: diagonal action of $\mathcal{T}_{\mathbb{N}}$ on $\mathcal{T}_{\mathbb{N}} \times \mathcal{T}_{\mathbb{N}}$...

- $\mathcal{T}_{\mathbb{N}}$: the full transformation monoid on $\mathbb{N} = \{0, 1, 2, \dots\}$.
- Elements: all mappings $\mathbb{N} \to \mathbb{N}$.
- Operation: composition: $x(\alpha\beta) = (x\alpha)\beta$.
- $\mathcal{T}_{\mathbb{N}}$ acts on itself via $\alpha \cdot \tau := \alpha\tau$.
- The direct square of this action: $\mathcal{T}_{\mathbb{N}}$ acts on $\mathcal{T}_{\mathbb{N}} \times \mathcal{T}_{\mathbb{N}}$ via

$$(\alpha, \beta) \cdot \tau := (\alpha\tau, \beta\tau).$$

. . . is monogenic (cyclic)

- Consider $\alpha, \beta \in \mathcal{T}_{\mathbb{N}}$ defined by:

$$x\alpha = 2x, \; x\beta = 2x + 1.$$

- Let $\gamma, \delta \in \mathcal{T}_{\mathbb{N}}$ be arbitrary.
- Define $\tau \in \mathcal{T}_{\mathbb{N}}$ by

$$x\tau := \begin{cases} t\gamma & \text{if } x = 2t \\ t\delta & \text{if } x = 2t + 1. \end{cases}$$

- Then:

$$x\alpha\tau = (2x)\tau = x\gamma,$$
$$x\beta\tau = (2x + 1)\tau = x\delta.$$

- I.e.: $(\gamma, \delta) = (\alpha, \beta)\tau$.

Different interpretations

- $\mathcal{T}_{\mathbb{N}} \times \mathcal{T}_{\mathbb{N}} = (\alpha, \beta)\mathcal{T}_{\mathbb{N}}$.
- The act $\mathcal{T}_{\mathbb{N}} \times \mathcal{T}_{\mathbb{N}}$ is cyclic (1-generated).
- This was first observed by Bulman-Fleming and McDowell (1989), and has applications for finite generation/presentability of wreath products of moinoids, and power monoids.
- $\mathcal{T}_{\mathbb{N}} \times \mathcal{T}_{N} \cong \mathcal{T}_{\mathbb{N}}$ (as acts).
- The right congruence on $\mathcal{T}_{\mathbb{N}}$ generated by (α, β) is the full congruence $\nabla_{\mathcal{T}_{\mathbb{N}}}$.
- Furthermore, any two $\gamma, \delta \in \mathcal{T}_{\mathbb{N}}$ are connected by an elementary (α, β)-sequence of length 1.

Right congruences

Definition

A right congruence on a semigroup S is any equivalence relation ρ which is also right compatible:

$$(\forall x, y, s \in S)((x, y) \in \rho \Rightarrow (xs, ys) \in \rho).$$

- Right congruences arise in representations of S by transformations (actions).
- S acts on the quotient S/ρ.
- When S is a monoid this coincides precisely with all monogenic representations.

Generation of congruences

Definition
Let S be a semigroup and $Y \subseteq S \times S$. The right congruence generated by Y is the smallest right congruence $\langle Y \rangle_{\mathrm{rc}}$ that contains Y.

Fact
We have $(a, b) \in \langle Y \rangle_{\mathrm{rc}}$ if and only if there exists an elementary sequence of the form

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ d_2 t_2 = c_3 t_3,\ \cdots,\ d_n t_n = b,$$

where $(c_i, d_i) \in Y \cup Y^{-1}$ and $t_i \in S^1$.

Remark
Think of this as 'connecting' a and b in a certain way.

Diameter

- S – a semigroup.
- $\nabla_S := S \times S$ – the full relation.
- Suppose $\nabla_S = \langle Y \rangle_{\text{rc}}$ for $|Y| < \infty$.
- $D(S, Y)$: the length of the longest Y-elementary sequence.
- $D(S)$: the minimum of all $D(S, Y)$ as Y ranges over finite generating sets for ∇_S.
- We call $D(S)$ the diameter of S.
- If S has a finite diameter we say it is psudofinite.
- Example: $\mathcal{T}_{\mathbb{N}}$ is pseudofinite, $D(\mathcal{T}_{\mathbb{N}}) = 1$.

Context (1)

- Pseudo-finite monoids were introduced by White (2017) to understand the relation between maximal ideals in semigroup algebras being finitely generated, and the algebra itself being finitely generated.
- Kobayashi (2007): ∇_S being finitely generated is equivalent to S being right FP_1.
- More detailed study: Gould, Quinn-Gregson, Zenab, Yang (2019).
- A group is pseudofinite iff it is finite.
- The same holds for: (weakly) left cancellative and for $\mathcal{L}^*$-simple semigroups; Gould, Miller, Quinn-Gregson, NR (2023).
- But there are also infinite semigroups with finite diameter, e.g. $\mathcal{T}_{\mathbb{N}}$.

Context (2)

- $D(S) = 1$ iff S has a finitely generated diagonal act.
- Finiteness condition: all finite semigroups have diameter 1.
- For a finitely generated semigroup: $D(S)$ is finite iff the graph obtained from the Cayley graph of S by ‘forgetting’ the directions has finite diameter.

Zeros and ideals

Proposition

If a monoid S has a zero element, then $D(S) \leq 2$.

Proof.

With respect to the generating set $\{(0,1)\}$ we have:

$$a = 1 \cdot a, 0 \cdot a = 0 = 0 \cdot b, 1 \cdot b = b,$$

and elementary sequence of length 2. ∎

Proposition

Let S be a monoid, and I an ideal of S. If I has finite diameter then

$$D(S) \leq D(I) + 2.$$

Guiding question

For some well known, natural semigroups of transformations, find their diameters.

Transformation semigroups

For an (infinite) set X:

- $\mathcal{PT}_X$: partial transformation monoid – all partial mappings $X \to X$.
- (Partial means not necessarily everywhere defined.)
- $\mathcal{T}_X := \{\alpha \in \mathcal{PT}_X : \text{dom}(\alpha) = X\}$: full transformation monoid.
- $\mathcal{I}_X := \{\alpha \in \mathcal{PT}_X : \alpha \text{ is 1-1}\}$: the symmetric inverse monoid.

Theorem

$D(\mathcal{PT}_X) = D(\mathcal{T}_X) = 1$, $D(\mathcal{I}_X) = 2$.

Proof

- $\mathcal{T}_X$, $\mathcal{PT}_X$ have cyclic diagonal acts.
- $\mathcal{I}_X$ has a zero element (empty map).

Injective transformations (1)

- $\mathcal{I}\mathsf{nj}_X := \mathcal{T}_X \cap \mathcal{I}_X$: the monoid of injective (full) mappings.
- $\mathcal{I}\mathsf{nj}_X$ has a minimal ideal.
- $\mathcal{BL}_X := \{\alpha \in \mathcal{I}\mathsf{nj}_X : |X \setminus X\alpha| = |X|\}$ – the Baer–Levy semigroup.
- $\mathcal{BL}_X$ is right cancellative and right simple (but not left).

Theorem (East, Gould, Miller, Quinn-Gregson, NR (2024))

$D(\mathcal{BL}_X) = 3.$

Hence $\mathcal{I}\mathsf{nj}_X$ has finite diameter $\leq 3 + 2 = 5$.

Theorem (East, Gould, Miller, Quinn-Gregson, NR (2024))

$D(\mathcal{I}\mathsf{nj}_X) = 4.$

Surjective transformations

- $\mathcal{S}\mathrm{urj}_X := \{\alpha \in \mathcal{T}_X : X\alpha = X\}$: the monoid of surjective mappings.
- $\mathcal{DBL}_X := \{\alpha \in \mathcal{S}\mathrm{urj}_X : |x\alpha^{-1}| = |X|, \forall x \in X\}$ – the dual Baer–Levy semigroup, the minimal ideal of $\mathcal{S}\mathrm{urj}_X$.

Theorem (East, Gould, Miller, Quinn-Gregson, NR (2024))

Neither $\mathcal{S}\mathrm{urj}_X$ nor $\mathcal{DBL}_X$ is pseudofinite (in fact, the full relation is not finitely generated for either).

Left-right

We can left-right dualise everything we have done so far!

- Left congruences.
- Left pseudofinite: the full congruence finitely generated as a left congruence.
- Left diameter: $D_l(S)$.
- $D(S) \to D_r(S)$.

Some results for the left diameter

East, Gould, Miller, Quinn-Gregson, NR (2024)

- $D_l(\mathcal{T}_X) = D_l(\mathcal{PT}_X) = 1$, $D_l(\mathcal{I}_X) = 2$.
- $\mathcal{I}\mathrm{nj}_X$, $\mathcal{BL}_X$ are not left-pseudofinite (in fact, the full relation is not finitely generated as a left congruence for either).
- $D_l(\mathcal{DBL}_X) = 2$.
- $D_l(\mathcal{S}\mathrm{urj}_X) \leq 2 + 2 = 4$.
- $D_l(\mathcal{S}\mathrm{urj}_X) = 4$.

Summary of results

Semigroup S	S^r f.g.?	∇_S f.g (right)?	$D_r(S)$	S^l f.g.?	∇_S f.g. (left)?	$D_l(S)$
$\mathbf{B}_X$	Yes	Yes	1	Yes	Yes	1
$\mathcal{PT}_X$	Yes	Yes	1	Yes	Yes	1
$\mathcal{I}_X$	Yes	Yes	2	Yes	Yes	2
$\mathcal{T}_X$	Yes	Yes	1	Yes	Yes	1
$\mathcal{S}_X$	Yes	No	n.a.	Yes	No	n.a.
$\mathcal{F}_X$	Yes	Yes	1	Yes	No	n.a.
$\mathcal{I}\mathrm{nj}_X$	Yes	Yes	4	Yes	No	n.a.
$\mathcal{BL}_{X,q}, q < \|X\|$	Yes	No	n.a.	No	No	n.a.
$\mathcal{BL}_X$	Yes	Yes	3	No	No	n.a.
$\mathcal{BL}_X^1$	Yes	Yes	3	Yes	No	n.a.
$\mathcal{S}_X \cup \mathcal{BL}_X$	Yes	Yes	4	Yes	No	n.a.
$\mathcal{S}\mathrm{urj}_X$	Yes	No	n.a.	Yes	Yes	4
$\mathcal{DBL}_{X,q}, q < \|X\|$	No	No	n.a.	Yes	No	n.a.
$\mathcal{DBL}_X$	No	No	n.a.	Yes	Yes	2
$\mathcal{DBL}_X^1$	Yes	No	n.a.	Yes	Yes	3
$\mathcal{S}_X \cup \mathcal{DBL}_X$	Yes	No	n.a.	Yes	Yes	4
$\mathcal{T}_X \setminus \mathcal{I}\mathrm{nj}_X$	No	No	n.a.	Yes	Yes	2
$\mathcal{T}_X \setminus \mathcal{S}\mathrm{urj}_X$	Yes	Yes	2	No	No	n.a.
$\mathcal{H}_X$	Yes	Yes	1	Yes	No	n.a.
$\mathcal{K}_X$	Yes	No	n.a.	Yes	No	n.a.
$\mathcal{P}_X$	Yes	Yes	1	Yes	Yes	1
$\mathcal{PB}_X$	Yes	Yes	1	Yes	Yes	1

Some questions

- Why are all numbers in the table small?
- Do there exist semigroups of arbitrarily large finite diameters?
- Gould & NR think YES: a construction.
- Are there natural examples of semigroups of arbitrarily large finite diameters?

Endomorphisms of chains...

- C: an (infinite) chain (totally ordered set).
- End(C): the endomorphism monoid of C (order preserving mappings $C \to C$).
- Endpoint: a max or min of C.
- Endpoint shiftable: C has an endpoint z and $C \setminus \{z\}$ is a quotient of C.

Examples

- No endpoint
  $\mathbb{Z}$: $\cdots < -2 < -1 < 0 < 1 < 2 < \dots$.
- Endpoint and endpoint shiftable
  $\mathbb{N}$: $1 < 2 < 3 < \dots$.
- Endpoint but not endpoint shiftable
  $\{-\infty\} \cup \mathbb{Z}$: $\infty < \cdots < -1 < 0 < 1 < \dots$.

. . . and their diameters

Theorem (East, Gould, Miller, Quinn-Gregson (2024))

For an infinite chain C:

- $D_r(\mathrm{End}(C)) = 2$ *if and only if C is endpoint shiftable. . .*
- *. . . and otherwise* $D_r(\mathrm{End}(C)) = 3$*;*
- $D_l(\mathrm{End}(C)) = 2$.

Some more questions

Question

Determine diameters of endomorphism monoids of other interesting infinite structures, e.g. the random graph, random tournament, countable atomless Boolean algebra, etc.

Question

Determine diameters of some monoids of continuous mappings of topological spaces: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{I}$rr, Cantor space, Urysohn space, etc.

Question

Can we describe semigroups for which every right congruence of finite index is finitely generated in a bounded way?

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THANK YOU!!!