# Semigroup actions and finiteness conditions


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PALS, CU Boulder, 7 November 2023

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# Outline

- Semigroups act on sets (representations by transformations).
- Actions $\leftrightarrow$ right (or left) congruences.
- Actions can be regarded as algebras in their own right (unary algebras).
- One can try to classify semigroups by the properties of their actions. (Analogous to rings and modules.)
- Finiteness condition: a property that is possessed by all finite semigroups.

# Outline (contd.)

- We discuss four finiteness conditions of semigroups defined by properties of their acts.
- Noetherian: every finitely generated $S$-act is finitely presented.
- Coherent: Every finitely generated subact of every finitely presented $S$-act is finitely presented.
- $FP_1$: The trivial $S$-act is finitely presented.
- Pseudofinite: the trivial $S$-act is finitely presented and there is a bound on derivation sequences.

# People/projects

Victoria Gould (York, UK), NR:
Right Noetherian and coherent monoids
(EPSRC funded project EP/V002953/1 + EP/V003224/1)

Postdocs: Matthew Brookes (St Andrews), Craig Miller (York)

Further collaborators: Tom Quinn-Gregson (York), James East (Western Sydney), ...


# Semigroups/monoids

- Monoid = semigroup $S$ with an identity element $1_S$ (meaning $x1_S = 1_S x = x, \forall x \in S$).
- $S$ is commutative if $xy = yx$ for all $x, y \in S$.
- $S$ is regular if for every $x \in S$ there exists $y \in S$ such that $xyx = x$.
- $S$ is inverse if for every $x \in S$ there exists a unique $y = x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.
- Every semigroup is isomorphic to a semigroup of mappings on a set. (Cayley Theorem for semigroups)
- Every inverse semigroup is isomorphic to an inverse semigroup of partial bijections on a set. (Cayley Theorem for inverse semigroups)

# Actions

### Definition

An action of a monoid $S$ on a set $X$ is a mapping $X \times S \to X$, $(x, s) \mapsto x \cdot s$, such that

$$(x \cdot s) \cdot t = x \cdot (st),\ x \cdot 1_S = x \quad (\forall x \in X; s, t \in S).$$

We also say: $S$ acts on $X$; $X$ is an $S$-act.

- $S$-acts $\leftrightarrow$ unary algebras (one operation per element of $S$).
- Subacts, homomorphisms, generators, presentations,...
- Occasionally we will talk about semigroup actions: omit $x \cdot 1_S = x$.

# Actions: examples

- $S$ acts on itself by multiplication: $x \cdot s = xs$. (Cayley representation)
- $S$ acts on any right ideal by multiplication.
- Right ideals are precisely the subacts of the $S$-act $S$.
- $S$ acts on $\{0\}$ via $0 \cdot s = 0$ (trivial $S$-act).

# Actions: generators, congruences

Let $S$ be a monoid, and $X$ an $S$-act.

- For $A \subseteq X$, the subact generated by $A$ is $A \cdot S := \{a \cdot s \,:\, a \in A, s \in S\}$.
- $X$ is finitely generated if $X = A \cdot S^1$ for some finite $A$.
- A congruence on $X$ is an equivalence relation $\sim$ such that

$$x \sim y \Rightarrow x \cdot s \sim y \cdot s \quad (x, y \in X,\ s \in S).$$

- A congruence on the $S$-act $S$ is called a right congruence of (the monoid) $S$.
- Right congruences of a group $G \leftrightarrow$ partitions of $G$ into right cosets of subgroups.

# Finitely presented actions

- For a relation $R$ on an $S$-act $X$, denote by $R^\sharp$ the congruence generated by $R$, i.e. the smallest congruence on $X$ containing $R$.
- For a set $B$, the free $S$-act over $B$ consists of $|B|$ disjoint copies of the $S$-act $S$; denote it by $F_B^S$.
- The $S$-act $S$ is the free monogenic (or cyclic) $S$-act.
- Right congruences on $S \leftrightarrow$ monogenic $S$-acts.
- An $S$-act $X$ is finitely presented if $X \cong F_B^S / R^\sharp$ for some finite $B$ and $R$.

# Finitely generated and presented actions: examples

### Example

The free $S$-act $F_B^S$ over finite $B$ is both finitely generated and finitely presented.

### Example

Every finite $S$-act $X$ is finitely generated. (by $X$)

### Example

If $S$ is finitely generated (as a monoid) then every finite $S$-act $X$ is finitely presented. (The multiplication table of elements of $X$ by the generators of $S$.)

### Example

If $S = A^*$ is the free monoid of countable rank then the trivial act $\{0\}$ is not finitely presented. (In any finite set of pairs, there would be an element of $A$ that doesn't appear.)

# (Right) noetherian monoids

### Definition

A semigroup $S$ is (right) noetherian if every finitely generated $S$-act is finitely presented.

### Theorem (standard)

*The following are equivalent:*

- *$S$ is noetherian.*
- *Every right congruence on $S$ is finitely generated.*
- *$S$ satisfies the ACC on right congruences.*

### Open Problem

Is every noetherian monoid finitely generated?

### Open Problem

Is the direct product of two noetherian monoids again noetherian?

# Noetherian monoids: generators

### Theorem

*All noetherian monoids belonging to the following classes are finitely generated:*

- *groups* [standard];
- *commutative semigroups* [Budach/Rédei 1963];
- *inverse semigroups* [Kozhukhov 1980; Miller, NR 2020].

### Remark

In fact for groups and inverse monoids the following holds: $S$ is noetherian iff all its subgroups/inverse subsemigroups are finitely generated. The analogue is not true for commutative semigroups: $\mathbb{N}_0 \times \mathbb{N}_0$ is noetherian but contains a non-finitely generated submonoid $(\mathbb{N} \times \mathbb{N})^1$.

### Conjecture (Brookes, Miller, et al.)

Every noetherian cancellative monoid is finitely generated.

# Noetherian monoids: direct products

### Theorem (Miller, NR 2020)

*Let $S$ be a finite or commutative or inverse monoid (including groups). If $S$ and $T$ are noetherian then so is $S \times T$.*

### Problem

*Develop a theory of congruences of direct products of monoids.*

### Remark

The direct product of noetherian semigroups need not be noetherian. Example: $\mathbb{N} \times \mathbb{N}$, which is commutative but not finitely generated.

# (Right) coherence

**Definition**

A monoid $S$ is (right) coherent if every finitely generated subact of any finitely presented $S$-act is finitely presented.

**Theorem (Gould 1992)**

*$S$ is coherent if and only if for every finitely generated right congruence $\rho$ and all $a, b \in S$, the following hold:*

(i) *the right annihilator congruence*

$$r(a\rho) := \{(s,t) \in S \times S \;:\; (as, at) \in \rho\}$$

*is finitely generated; and*

(ii) *the subact $(a\rho) \cdot S \cap (b\rho) \cdot S$ of the $S$-act $S/\rho$ is finitely generated.*

# Coherent monoids: examples

**Theorem**

*All of the following monoids are coherent:*

- *finite* [obvious];
- *groups* [easy];
- *finitely generated commutative* [Carson, Gould 2021];
- *free* [Gould, NR 2017];
- *free left ample* [Gould, Hartmann 2017].

*The free inverse monoid of rank* $\geq 2$ *is not coherent* (ibid.).

# Non-coherence of natural transformation monoids

**Theorem (Brookes, Gould, NR, in prep.)**

*If $A$ is an infinite set, then none of the following are coherent:*

- *the full transformation monoid $T_A$;*
- *the partial transformation monoid $PT_A$;*
- *the symmetric inverse monoid $I_A$;*
- *the partition monoid $P_A$.*

**Remark**

Behind all those results is a technical condition in terms of elements and left/right ideals, which all the above monoids happen to satisfy.

# Coherence and direct products

### Theorem (Dandan, Gould, Hartmann, NR, Zenab 2020)

*If $S$ is a finite monoid, and $T$ a coherent monoid then $S \times T$ is coherent.*

### Theorem (ibid.)

*The direct product $F_3 \times F_3$ of two free monoids of rank 3 is not coherent.*

### Questions

- When is the direct product of two coherent monoids coherent?
- When is the direct product $F_m \times F_n$ of two free monoids coherent?
- Is the direct product $F_1 \times F_n$ coherent? (Conj: yes.)
- Is it true that the direct product of a group and a coherent monoid is coherent?

# Finitely presented trivial act $\mathrm{FP}_1$

## Definition

A monoid $S$ has the property $\mathrm{FP}_1$ if the trivial $S$-act is finitely presented.

## Facts

- $\mathrm{FP}_1 \Leftrightarrow$ the full relation $\nabla := S \times S$ is finitely generated as a right congruence of $S$ [standard].
- A group satisfies $\mathrm{FP}_1$ iff it is finitely generated [standard].

Y. Dandan, V. Gould, T. Quinn-Gregson, R.-E Zenab, Semigroups with finitely generated universal left congruence, Monatsh. Math. 190 (2019), 689–724.

# Generation of congruences; $R$-sequences

## Definition

An *$R$-sequence* (for $R \subseteq S \times S$):

$$s = a_1 u_1, b_1 u_1 = a_2 u_2, b_2 u_2 = a_3 u_3, \ldots, b_{n-1} u_{n-1} = a_n u_n, b_n u_n = t,$$

where $(a_i, b_i) \in R$, $u_i \in S$.

## Proposition

*For $R \subseteq S \times S$, $s, t \in S$:*

$$(s, t) \in R^\sharp \iff \exists \text{ an } R\text{-sequence from } s \text{ to } t.$$

# Diameter, pseudofiniteness

**Definition**

Let $S$ be an $\mathrm{FP}_1$ monoid, and let $R \subseteq S \times S$ be a finite generating set for $\nabla$. The distance between $s, t \in S$ with respect to $R$ is

$$d_R(s,t) := \min\{n \in \mathbb{N} \,:\, \exists \text{ an } R-\text{sequence from } s \text{ to } t \text{ of length } n\}.$$

The diameter of $S$ with respect to $R$ is

$$D(S,R) := \sup\{d_R(s,t) \,:\, s,t \in S\}.$$

The diameter of $S$ is

$$D(S) := \min\{D(S,R) \,:\, \langle R\rangle = \nabla\}.$$

If $D(S)$ is finite, we say that $S$ is pseudofinite.

# Pseudofinite vs. finite vs. minimal ideal

## Facts

- Finite $\Rightarrow$ pseudofinite.
- For groups: finite $\Leftrightarrow$ pseudofinite.
- True in general?
- No: If $S$ has a zero element then it is pseudofinite.
  ($\nabla$ is generated by $(1_S, 0_S)$ and $D(S) \leq 2$.)
- More generally: If $S$ has a finite (minimal) ideal then $S$ is pseudofinite.
- Question: Does every pseudofinite monoid have a minimal ideal?

# Pseudofiniteness and minimal ideals (+)

## Theorem

- *Every pseudofinite commutative monoid has a finite minimal ideal (which is then necessarily a group)* [Gould, Miller, Quinn-Gregson, NR 2023]*.*
- *Every pseudofinite inverse monoid has a finite minimal ideal (which is then necessarily a group)* [Dandan, Gould, Quinn-Gregson, Zenab 2019]*.*
- *Every pseudofinite completely regular monoid has a finite minimal ideal (which is then necessarily a completely simple semigroup)* (ibid.)*.*
- *. . . more. . .* [Gould, Miller, Quinn-Gregson, NR 2023]*.*

# Pseudofiniteness and minimal ideals $(-)$

## Theorem (Gould, Miller, Quinn-Gregson, NR 2023)

*Let $X$ be an infinite set, and let $X = \bigcup_{i\in\mathbb{N}} X_i$ be a partition into subsets of size $|X|$. Then*

$$\mathcal{U}_X = \{\alpha : X \to X \mid X_i\alpha \subseteq \bigcup_{j\geq i} X_j,\ \forall i \in \mathbb{N}\}$$

*is a pseudofinite monoid with no minimal ideal.*

## Theorem (ibid.)

*There exists a pseudofinite monoid without minimal ideal, which has any one of the following properties: (a) regular; (b) $\mathcal{J}$-trivial; (c) periodic.*

## Remark

The proof relies on an explicit construction, which is an extension of a Rees matrix semigroup construction.

# Small diameter

J. East, V. Gould, C. Miller, T. Quinn-Gregson, NR, in prep.: many natural monoids of transformations and partitions are pseudofinite, and they all have small diameter.

## Theorem

*Let $A$ be an infinite set. All of the following are pseudofinite, with the diameter as stated:*

- *the full transformation monoid $T_A$, with $D(T_A) = 1$;*
- *the partial transformation monoid $PT_A$, with $D(PT_A) = 1$;*
- *the symmetric inverse monoid $I_A$, with $D(I_A) = 2$;*
- *the monoid $F_A$ of finite-to-one mappings, with $D(F_A) = 1$;*
- *the monoid $Inj_A$ of all injective mappings, with $D(Inj_A) = 4$;*
- *the Baer-Levi semigroup $BL_A$, with $D(Inj_A) = 3$;*
- *...more...*

# Pseudofiniteness: some questions

## Question

- Do there exist monoids of larger finite diameters?
- Yes: Gould, NR, work in progress; arbitrarily large diameters; via a construction.
- Do there exist 'natural' examples?

## Question

- Fact: $D(S) = 1$ iff $S \times S$ considered as an $S$-act is finitely generated.
- Does there exist a similar (necessarily more complicated) characterisation for $D(S) = n$ for $n = 2, 3, \ldots$?

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