Eliminating ~~algebraicity~~ and enforcing injectivity in infinite-domain constraint satisfaction

Moritz Schöbi, TU Wien

joint work with M. Pinsker, J. Rydval & C. Spiess

Panglobal Algebra and Logic Seminar, Spring '25

Eliminating ~~algebraicity~~ and enforcing injectivity in infinite-domain constraint satisfaction

Content

- Setting the scene
- The Bodirsky-Pinsker conjecture
- Three questions
- Short answers
- Digging deeper - the construction

Setting the scene

$\mathbb{A} = (A; (R_i)_{i \in I})$ relational structure with finite signature τ,
at most countable

CSP($\mathbb{A}$):

Input: finite τ-structure $\mathbb{C}$

Decide: $\mathbb{C} \to \mathbb{A}$?

Example: 3-COLORING = CSP($\mathbb{K}_3$) NP-complete

$\mathbb{A}$ pp-constructs $\mathbb{B}$ if $\mathbb{B}$ is hom. eq. to sth. pp-definable in a finite power of $\mathbb{A}$.
$\rightsquigarrow$ CSP($\mathbb{B}$) reduces to CSP($\mathbb{A}$).

$$\mathrm{Pol}(\mathbb{A}) := \bigcup_{n \geq 1} \mathrm{Hom}(\mathbb{A}^n, \mathbb{A})$$

Setting the scene

Theorem (Bulatov & Zhuk '17)
$\mathbb{A}$ finite relational structure. Then either
i) $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ and $CSP(\mathbb{A})$ is NP-complete, or
ii) $Pol(\mathbb{A})$ satisfies the Siggers-identity

$$s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y),$$

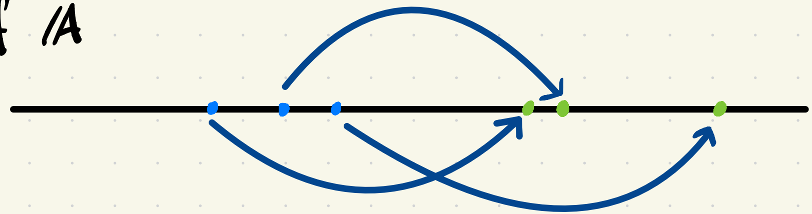
and $CSP(\mathbb{A})$ is polynomial-time solvable.

Big goal: lift this (as far as possible) to infinite-domain CSPs

The Bodirsky-Pinsker conjecture

$\mathbb{A}$ is...

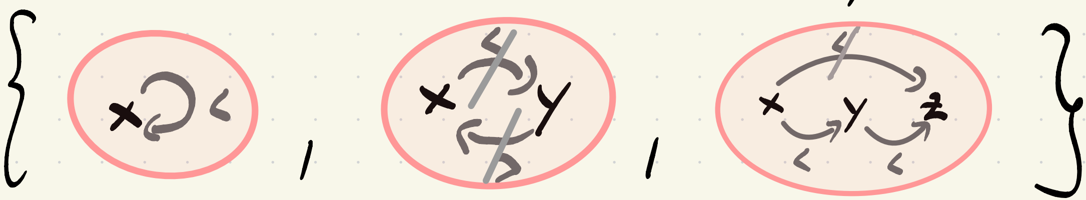
homogeneous if every isomorphism between finite substructures extends to an automorphism of $\mathbb{A}$



$(\mathbb{Q},<)$

finitely bounded if $\exists\, \mathcal{N}$ finite set of finite τ-structures s.t. for all finite τ-str. $\mathbb{B}$:

$$\mathbb{B} \hookrightarrow \mathbb{A} \iff \forall \mathbb{M} \in \mathcal{N} \quad \mathbb{M} \not\hookrightarrow \mathbb{B}$$



The Bodirsky-Pinsker conjecture

Conjecture (Bodirsky + Pinsker '11):
$\mathbb{A}$ reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Then one of the following holds:

i) $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ ($\to$ CSP($\mathbb{A}$) is NP-complete)

ii) Pol($\mathbb{A}$) satisfies the pseudo-Siggers identity

$$\alpha \circ s(x,y,x,z,y,z) \approx \beta \circ s(y,x,z,x,z,y),$$

and CSP($\mathbb{A}$) is polynomial-time solvable.

Note: Siggers identity became weaker pseudo-variant.

The three questions

Conjecture (Bodirsky + Pinsker '11):
$\mathbb{A}$ reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Then one of the following holds:

i) $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ ($\to$ CSP($\mathbb{A}$) is NP-complete)

ii) Pol($\mathbb{A}$) satisfies the pseudo-Siggers identity $\alpha \circ s(x,y,x,z,y,z) = \beta \circ s(y,x,z,x,z,y)$, and CSP($\mathbb{A}$) is polynomial-time solvable.

1. Can we impose additional structural assumptions on the structures in scope of the conjecture w.l.o.g.?
2. Can we impose significant algebraic assumptions on the polymorphisms of these structures w.l.o.g.?
3. Are there algorithmic links between CSPs from the conjecture and PCSPs?

Short answer 1

Conjecture (Bodirsky + Pinsker '11):
$\mathbb{A}$ reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Then one of the following holds:

i) $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ ($\to CSP(\mathbb{A})$ is NP-complete)

ii) $Pol(\mathbb{A})$ satisfies the pseudo-Siggers identity $\alpha \circ s(x,y,x,z,y,z) = \beta \circ s(y,x,z,x,z,y)$, and $CSP(\mathbb{A})$ is polynomial-time solvable.

Q1: Can we impose additional structural assumptions on the structures in scope of the conjecture w.l.o.g.?

A1: Find $\mathbb{A}^*$ red. of fb. hom. $\widetilde{\mathbb{B}}$ st.

- nice properties of $\mathbb{A}, \mathbb{B}$ transfer
- $\mathbb{A}^*$ pp-c $\mathbb{K}_3 \Leftrightarrow \mathbb{A}$ pp-c $\mathbb{K}_3$
- $CSP(\mathbb{A})$ and $CSP(\mathbb{A}^*)$ efficiently interreducible
- $\mathbb{A}^*$ has no algebraicity

Short answer 2

Conjecture (Bodirsky + Pinsker '11):

$\mathbb{A}$ reduct of a finitely bounded homogeneous structure $\mathbb{B}$. Then one of the following holds:

i) $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ ($\rightarrow$ CSP($\mathbb{A}$) is NP-complete)

ii) Pol($\mathbb{A}$) satisfies the pseudo-Siggers identity $\alpha \circ s(x,y,x,z,y,z) = \beta \circ s(y,x,z,x,z,y)$, and CSP($\mathbb{A}$) is polynomial-time solvable.

Q2: Can we impose significant algebraic assumptions on the polymorphisms of these structures w.l.o.g.?

A2: Can expand $\mathbb{A}^*$ and $\widetilde{\mathbb{B}}$ from A1 by a relation I_4 s.t. $\mathbb{A}^*_{I_4}$ only has essentially injective polymorphisms, while other properties are preserved.

$\rightsquigarrow$ identities characterizing tractability must be injective!

Promise Constraint Satisfaction Problems

$\mathbb{S}_1, \mathbb{S}_2$ finite τ-structures s.t. $\mathbb{S}_1 \to \mathbb{S}_2$

$\boxed{\begin{array}{l} \mathrm{PCSP}(\mathbb{S}_1, \mathbb{S}_2): \\ \text{Input: Finite } \tau\text{-structure } \mathbb{C} \\ \text{Decide: } \mathbb{C} \to \mathbb{S}_1 \text{ or } \mathbb{C} \not\to \mathbb{S}_2 \end{array}}$

"promise": all input structures fall into one of these categories

Note: $\mathrm{PCSP}(\mathbb{S}_1, \mathbb{S}_1) = \mathrm{CSP}(\mathbb{S}_1)$

Example: Approximate graph coloring, e.g. $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

$\mathbb{A}$ is a **sandwich** of $(\mathbb{S}_1, \mathbb{S}_2)$ if $\mathbb{S}_1 \to \mathbb{A} \to \mathbb{S}_2$

$\leadsto$ solve $\mathrm{CSP}(\mathbb{A})$ to solve $\mathrm{PCSP}(\mathbb{S}_1, \mathbb{S}_2)$

Promise Constraint Satisfaction Problems

$\mathbb{S}_1, \mathbb{S}_2$ finite τ-structures s.t. $\mathbb{S}_1 \to \mathbb{S}_2$

PCSP($\mathbb{S}_1, \mathbb{S}_2$):
Input: Finite τ-structure $\mathbb{C}$
Decide: $\mathbb{C} \to \mathbb{S}_1$ or $\mathbb{C} \not\to \mathbb{S}_2$

$\mathbb{A}$ sandwich of $(\mathbb{S}_1, \mathbb{S}_2)$ if $\mathbb{S}_1 \to \mathbb{A} \to \mathbb{S}_2$

Q: Are there finite PCSP templates with no finite tractable sandwich (not finitely tractable), but an infinite one?

Barto '19: Yes! But: Example not in scope of BP-conjecture.

Q: ω-cat. example? Example in scope of conjecture?

Short answer 3

Q3: Are there algorithmic links between CSPs from the conjecture and PCSPs?

A3: Theorem (Pinsker + Rydval + S. + Spiess '25)

$\mathbb{A}$ red. of $\mathbb{B}$ fin. bd. hom. Find $\mathbb{A}'$ red. of fin. bd. hom $\mathbb{B}'$ and a PCSP template $(\mathbb{S}_1, \mathbb{S}_2)$ s.t.

- nice properties of $\mathbb{A}, \mathbb{B}$ transfer
- $\mathbb{A}'$ pp-c $\mathbb{K}_3$ $\Leftrightarrow$ $\mathbb{A}$ pp-c $\mathbb{K}_3$
- CSP($\mathbb{A}$) and CSP($\mathbb{A}'$) efficiently interreducible
- $\mathbb{A}'$ is sandwich for $(\mathbb{S}_1, \mathbb{S}_2)$
- $(\mathbb{S}_1, \mathbb{S}_2)$ not finitely tractable

Digging deeper: structural notions

$\mathbb{A}$ is ...

CSP-injective if for all finite τ-str. $\mathbb{B}$:

$\mathbb{B} \to \mathbb{A} \iff \mathbb{B} \to \mathbb{A}$ injectively $(\mathbb{Q},<)$, RG

ω-categorical if for all n, $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has only finitely many orbits $(\mathbb{Q},<)$, RG

a model-complete core if all endomorphisms are embeddings $(\mathbb{Q},<)$, $\mathbb{K}_\omega$

$\mathbb{A}$ has no algebraicity if for all n, $\bar{a} \in A^n$, $\mathrm{Aut}(\mathbb{A}/\bar{a})$ does not stabilize any $a' \notin \bar{a}$. $(\mathbb{Q},<)$, RG

Digging deeper: Datalog

Datalog extends existential positive f.o. logic with formation rules whose semantics is specified using inflationary fixed points.

Example:

$$x \sqsubset y \leftarrow x < y$$
$$x \sqsubset y \leftarrow \exists z \; x \sqsubset z \wedge z \sqsubset y$$

$\mathrm{ifp}_{\sqsubset}(x,y)$ ... fixed point

$\rightsquigarrow$ inflationary f.p. is transitive closure of $<$

Datalog **solves** $CSP(\mathbb{A})$ if there is a Datalog sentence identifying NO-instances.

Note:
- Example solves $CSP(\mathbb{Q}, <)$ via $\exists x \; \mathrm{ifp}_{\sqsubset}(x, x)$
- Datalog $\sim$ bounded width
- Evaluation of Datalog formulas runs in polynomial time

Digging deeper: Datalog II

$\mathbb{A}$ σ-structure, $\mathbb{B}$ τ-structure

- CSP($\mathbb{A}$) Datalog-reduces to CSP($\mathbb{B}$) if there is a mapping I: fin σ-str. $\to$ fin τ-str. s.t. $\mathbb{C} \to \mathbb{A} \iff I(\mathbb{C}) \to \mathbb{B}$, and $I(\mathbb{C})$ can be defined from $\mathbb{C}$ using Datalog formulas.

- If CSP($\mathbb{B}$) is in complexity class containing Datalog, e.g. P, so is CSP($\mathbb{A}$).

- Datalog reductions $\supsetneq$ pp-constructions

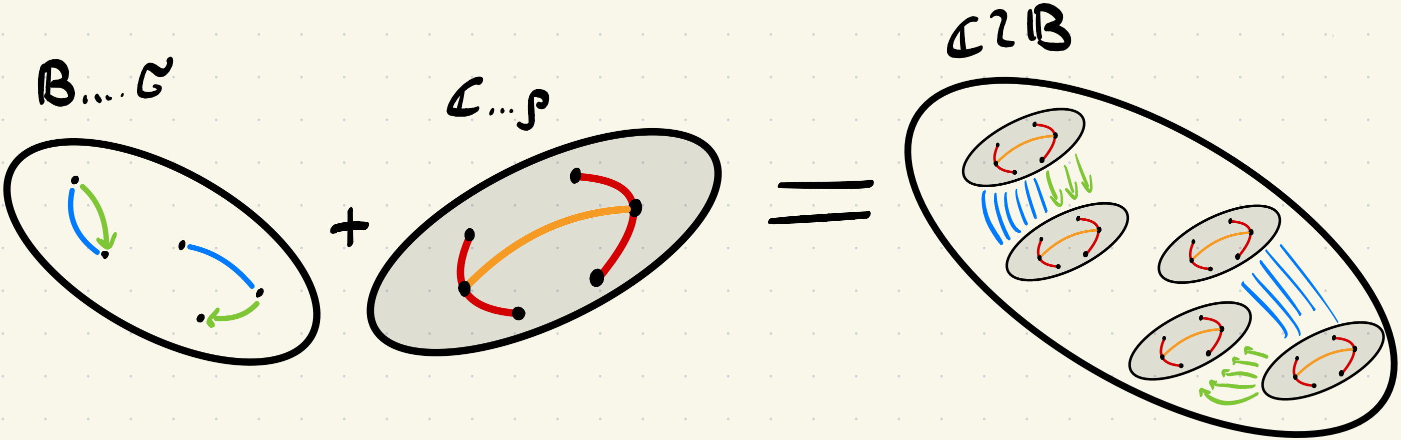
Digging deeper: Question 1

1. Can we impose additional structural assumptions on the structures in scope of the conjecture w.l.o.g.?

Want: no algebraicity + CSP-inj while preserving nice properties.

$\rightsquigarrow$ Blow up structures in a clever way!

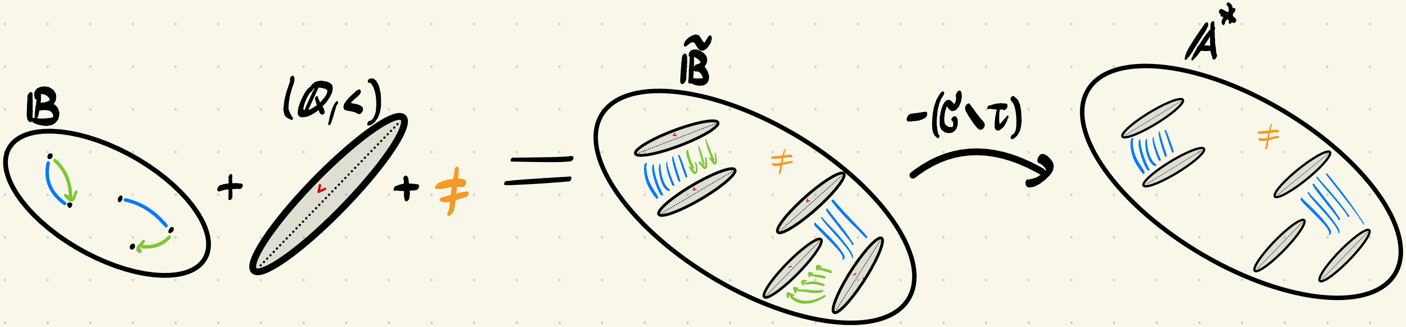
Digging deeper: The wreath product



- Replace elements of $\mathbb{B}$ with copies of ρ-structure $\mathbb{C}$
- $\mathbb{C} \wr \mathbb{B}$ has sign. $\sigma \cup \rho \cup \{E\}$, E is equiv. rel "$\mathbb{C}$-bubbles"
- ρ-relations within bubbles, σ-relations between bubbles
- $\mathrm{Aut}(\mathbb{C} \wr \mathbb{B}) \cong \mathrm{Aut}(\mathbb{C}) \wr \mathrm{Aut}(\mathbb{B})$
- $\mathbb{C} \wr \mathbb{B}$ inherits many nice properties of $\mathbb{B}$, $\mathbb{C}$

Digging deeper: the blowup

- Let $\mathbb{A}$ τ-reduct of $\mathbb{B}$ fin. bd. hom
- $\tilde{\mathbb{B}} := (\mathbb{Q},<) \wr \mathbb{B}$ + global $\neq$ has no algebraicity
- $\mathbb{A}^* := \tau \cup \{\neq, E\}$-reduct of $\tilde{\mathbb{B}}$
- $\mathbb{A}^*$ is CSP-injective
- $\mathbb{A}^*$ pp-c $\mathbb{K}_3$ iff $\mathbb{A}$ does
- CSP($\mathbb{A}^*$) and CSP($\mathbb{A}$) are Datalog-interreducible



Digging deeper: Answer 1

Can we impose additional structural assumptions on the structures in scope of the conjecture w.l.o.g.?

Theorem (Pinsker + Rydval + S. + Spiess '25)

$\mathbb{A}$ non-trivial reduct of $\mathbb{B}$ fin. bd. hom. Then $\exists\, \mathbb{A}^*$ reduct of $\widetilde{\mathbb{B}}$ fin. bd. hom. without algebraicity st.

- $\mathbb{A}^*$ is CSP-injective
- CSP($\mathbb{A}^*$) and CSP($\mathbb{A}$) are Datalog-interreducible
- $\mathbb{A}^*$ is model-complete core iff $\mathbb{A}$ is
- $\widetilde{\mathbb{B}}$ is Ramsey iff $\mathbb{B}$ is
- $\mathbb{A}^*$ pp-constructs $\mathbb{K}_3$ iff $\mathbb{A}$ does

Digging deeper: Question 2

Can we impose significant algebraic assumptions on the polymorphisms of these structures w.l.o.g.?

$\mathbb{A}$ is Pol-injective if all polymorphisms of $\mathbb{A}$ are essentially injective.

Facts: - $\mathbb{A}$ is Pol-injective iff $\mathrm{Pol}(\mathbb{A})$ preserves

$$I_4 := \{(x,y,u,v) \mid x = y \to u = v\}$$

- adding I_4 to a CSP-inj. structure yields Datalog-interreducible CSPs (but we lose CSP-inj.)

⇝ By adding I_4 to $\tilde{\mathbb{B}}$ from Question 1, we get...

Digging deeper: Answer 2

Can we impose signi-icant algebraic assumptions on the polymorphisms of these structures w.l.o.g.?

Theorem (Pinsker + Rydval + S. + Spiess '25)

$\mathbb{A}$ non-trivial reduct of $\mathbb{B}$ fin. bd. hom. Then $\exists$ $\mathbb{A}^*_{I_4}$ reduct of $\tilde{\mathbb{B}}_{I_4}$ fin. bd. hom. without algebraicity st.

- $\mathbb{A}^*_{I_4}$ is Pol-injective
- $CSP(\mathbb{A}^*_{I_4})$ and $CSP(\mathbb{A})$ are Datalog-interreducible
- $\mathbb{A}^*_{I_4}$ is model-complete core iff $\mathbb{A}$ is
- $\tilde{\mathbb{B}}_{I_4}$ is Ramsey iff $\mathbb{B}$ is
- $\mathbb{A}^*_{I_4}$ pp-constructs $\mathbb{K}_3$ iff $\mathbb{A}$ does

$\leadsto$ Identities characterizing tractability in conjecture must be (essentially) injective! E.g. $\alpha \circ s(x,y,x,z,y,z)$
$= \beta \circ s(y,x,z,x,z,y)$

Thank you!