Spectra of BCK-algebras

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Overview

In my dissertation I proved that the prime spectrum of any commutative BCK-algebra is nearly a spectral space (to be made precise later). A natural question is: what happens without the assumption of commutativity? I worked on this briefly (and even spoke at BLAST 2021), but put the project aside until recently when I started thinking about spectra of finite BCK-algebras.

Many of the results I'll discuss today are contained, in a much more general setting, in the paper:

- *Abstractly constructed prime spectra*
  Facchini, Alberto; Finocchiaro, Carmelo Antonio; Janelidze, George
  Algebra Universalis **83**, 1 (2022), Article No. 8, 38 pp.

Definitions

Definition

A *BCK-algebra* is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of type $(2, 0)$ such that

1. $\big[(x \cdot y) \cdot (x \cdot z)\big] \cdot (z \cdot y) = 0$
2. $\big[x \cdot (x \cdot y)\big] \cdot y = 0$
3. $x \cdot x = 0$
4. $0 \cdot x = 0$
5. $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

for all $x, y, z \in A$.

These algebras are partially ordered by: $x \leq y$ iff $x \cdot y = 0$.

A BCK-algebra $\mathbf{A}$ is *bounded* if there is an element $1 \in A$ such that $x \cdot 1 = 0$ for all $x \in A$.

Definitions

The element $x \wedge y := y \cdot (y \cdot x)$ is a lower bound for x and y, but may not be the greatest lower bound.

However, if $x \wedge y = y \wedge x$, then $x \wedge y$ is the greatest lower bound.

If $x \wedge y = y \wedge x$ for all $x, y \in A$, we say $\mathbf{A}$ is *commutative*.

Proposition (Iséki & Tanaka, 1978)

A BCK-algebra $\mathbf{A}$ is commutative if and only if it is a semilattice with respect to $\wedge$.

However, it can happen that the underlying poset of $\mathbf{A}$ is a lower semilattice even if $\mathbf{A}$ is not commutative. We call these *BCK-semilattices*.

Examples

1. Any Boolean algebra **B** is a bounded commutative BCK-algebra via $x \cdot y = x \wedge (\neg y)$.

2. $\mathbf{R} = \langle \mathbb{R}_{\geq 0}; \cdot, 0 \rangle$ is a commutative BCK-algebra with operation $x \cdot y := \max\{x - y, 0\}$. It is not bounded.

   $\mathbb{N}_0$ is a subalgebra of **R**

   Any finite chain $\mathbf{n} = \{0, 1, \ldots, n-1\}$ is a subalgebra of **R**

3. Let X be any set and **A** any BCK-algebra. Then the set of functions

$$\mathbf{F}(X, \mathbf{A}) = \{ f : X \to \mathbf{A} \}$$

   is a BCK-algebra with pointwise operation:

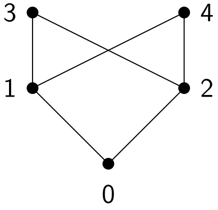
$$(f \cdot g)(x) = f(x) \cdot_{\mathbf{A}} g(x) .$$

Examples

4. An example of a non-commutative BCK-*lattice* is given by $\mathbb{N}_0$ with the operation

$$x \cdot y = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{if } x > y \neq 0 \\ x & \text{if } x > y = 0\,. \end{cases}$$

5. And of course, not all BCK-algebras are BCK-semilattices. The smallest example has order 5:



Ideals

Given a BCK-algebra $\mathbf{A}$, a subset I is an *ideal* if

1. $0 \in I$
2. if $x \cdot y \in I$ and $y \in I$, then $x \in I$.

Let $\mathrm{Id}(\mathbf{A})$ denote the lattice of ideals of $\mathbf{A}$.

An ideal I of $\mathbf{A}$ is *irreducible* if it is meet-irreducible as an element of $\mathrm{Id}(\mathbf{A})$.

An ideal I of $\mathbf{A}$ is *meet-prime* if it is meet-prime as an element of $\mathrm{Id}(\mathbf{A})$.

Let $\mathbf{A}$ be a BCK-semilattice. Define a proper ideal P to be *prime* if $\mathrm{glb}\{x, y\} \in P$ implies $x \in P$ or $y \in P$.

If $\mathbf{A}$ is commutative, this looks like: $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$.

Ideals

Theorem (Pałasinski, 1981)

In a BCK-semilattice, the following are equivalent:

1. *P is an irreducible ideal,*
2. *P is a meet-prime ideal,*
3. *P is a prime ideal.*

Note: (1) and (2) are equivalent for *any* BCK-algebra since $\mathsf{Id}(\mathbf{A})$ is a distributive lattice (Pałasinski, 1981).

Let $\mathrm{X}(\mathbf{A})$ denote the set of irreducible ideals of $\mathbf{A}$.

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

We will write $\sigma(a)$ for $\sigma(\{a\})$.

Proposition

(1)The family $\mathcal{T}(\mathbf{A}) = \left\{ \sigma(I) \mid I \in \mathrm{Id}(\mathbf{A}) \right\}$ is a topology on $\mathrm{X}(\mathbf{A})$

(2) $\mathcal{T}_0(\mathbf{A}) = \left\{ \sigma(a) \mid a \in A \right\}$ is a basis for this topology.

The space $\left(\mathrm{X}(\mathbf{A}) \, , \, \mathcal{T}(\mathbf{A}) \right)$ is the **spectrum** of **A**.

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The first proof seems to be a result of Aslam, Deeba, and Thaheem in 1992.

In their proof, they assume **A** is commutativity, but if you look closely what they're really using is the fact that prime ideals (in a comm. BCK-alg.) are meet-prime.

So in fact, this is actually true for all BCK-algebras (since we are taking $\mathrm{X}(\mathbf{A})$ to be the irreducible $\equiv$ meet-prime ideals).

A *spectral space* is a topological space X that is homeomorphic to the prime spectrum of a commutative ring.

Hochster famously characterized spectral spaces in his PhD thesis:

Theorem (Hochster, 1969)

A topological space X is spectral iff

- H1 *X is compact*
- H2 *X is T_0*
- H3 *the set $\mathcal{K}^{\circ}(X)$ of compact open subsets is a basis that is closed under finite intersections, and*
- H4 *X is sober: every irreducible closed subset is the closure of a unique point.*

Theorem (Meng & Jun, 1998)

If $\mathbf{A}$ is a bounded commutative BCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a spectral space.

What happens if **A** is commutative but not necessarily bounded?

Boundedness is sufficient to show $\mathrm{X}(\mathbf{A})$ is compact, but certainly not necessary: if **A** has finitely many ideals then $\mathrm{X}(\mathbf{A})$ is a finite space and hence compact.

Theorem (E., 2020)

Let **A** *be a commutative (not necessarily bounded) BCK-algebra.*

1. $\mathrm{X}(\mathbf{A})$ *is locally compact*
2. $\mathrm{X}(\mathbf{A})$ *is* T_0
3. $\mathcal{K}^\circ\mathrm{X}(\mathbf{A})$ *is a basis that is closed under finite intersections*
4. $\mathrm{X}(\mathbf{A})$ *is sober*

That is, $\mathrm{X}(\mathbf{A})$ is a locally compact *generalized spectral space*.

Proposition (E., 2020)

Let **A** be a commutative (not necessarily bounded) BCK-algebra. Then $\mathrm{X}(\mathbf{A})$ is compact (and thus spectral) iff **A** is finitely generated as an ideal.

But looking closely at the proof.... I never use the assumption of commutativity!

If one replaces all instances of "prime ideal" with "irreducible ideal," the proof goes through otherwise unchanged.

Question

What other results still work without commutativity?

One thing we do need to know is that the basic open sets $\sigma(a)$ are compact.

In my previous work, I cited a proof by Meng & Jun which uses both commutativity *and* boundedness in a fundamental way.

There is an easy proof for any BCK-algebra $\mathbf{A}$:

Lemma

For each $a \in \mathbf{A}$, the set $\sigma(a)$ is compact open in $\mathrm{X}(\mathbf{A})$.

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Proof.

Let $\{\sigma(I_\lambda)\}_{\lambda\in\Lambda}$ be an open cover of $\sigma(a)$.
Then

$$\sigma(a) \subseteq \bigcup_{\lambda\in\Lambda} \sigma(I_\lambda) = \sigma\Big(\bigvee_{\lambda\in\Lambda} I_\lambda\Big)$$

and one can show that $a \in \bigvee_{\lambda\in\Lambda} I_\lambda$. By some general BCK-theory, this means there are $b_1, b_2, \ldots, b_n \in \bigcup_{\lambda\in\Lambda} I_\lambda$ such that

$$(\cdots((ab_1)b_2)\cdots b_{n-1})b_n = 0\,.$$

Let us say $b_i \in I_{\lambda_i}$ for $i = 1, \ldots n$. Then in fact, $a \in \bigvee_{i=1}^n I_{\lambda_i}$, and therefore $\sigma(a) \subseteq \sigma\Big(\bigvee_{i=1}^n I_{\lambda_i}\Big) = \bigcup_{i=1}^n \sigma(I_{\lambda_i})\,.$ □

Other results that still work without commutativity:

Proposition

1. $\mathrm{X}(\mathbf{A})$ is locally compact
2. $\mathcal{K}^{\circ}\mathrm{X}(\mathbf{A})$ is a basis
3. $\mathrm{X}(\mathbf{A})$ is sober

The first two are straightforward, the third requires a little more work.

The key ingredients for (3) are:

- σ is injective, which can be proved without commutativity.
- $\mathrm{X}(\mathbf{A})$ consists of *irreducible* ideals.
- Every ideal I is the intersection of the irreducible ideals containing I, which can be proved without commutativity.

What about closure of $\mathcal{K}^\circ\mathrm{X}(\mathbf{A})$ under finite intersections?

Given $U, V \in \mathcal{K}^\circ\mathrm{X}(\mathbf{A})$, we have

$$U \cap V = \left(\bigcup_{i=1}^{n} \sigma(a_i)\right) \cap \left(\bigcup_{k=1}^{m} \sigma(b_k)\right) = \bigcup_{i=1}^{n}\bigcup_{k=1}^{m} \sigma(a_i) \cap \sigma(b_k)$$

$$= \underbrace{\bigcup_{i=1}^{n}\bigcup_{k=1}^{m} \sigma(a_i \wedge b_k)}_{\text{if } \mathbf{A} \text{ is comm.}}$$

$$= \underbrace{\bigcup_{i=1}^{n}\bigcup_{k=1}^{m} \sigma\left(\mathrm{glb}\{a_i, b_k\}\right)}_{\text{if } \mathbf{A} \text{ is a BCK-semilattice}}$$

Proposition (E.)

If **A** is a BCK-semilattice, then $\mathrm{X}(\mathbf{A})$ is locally compact generalized spectral space. Further, $\mathrm{X}(\mathbf{A})$ is spectral iff **A** is finitely generated as an ideal.

But, in general, the intersection of two compact open subsets need not be compact.

Of course, an open set $\sigma(I)$ is compact iff I is a finitely generated ideal, which motivates the question:

Question

What conditions on a BCK-algebra gaurantee that the intersection of two finitely generated ideals is itself finitely generated?

However, if **A** is only has finitely many ideals, this problem resolves itself.

Spectral stats for finite BCK-algebras

Meng & Jun's book on BCK-algebras contains all BCK-algebras up to order 5, including a breakdown of their ideal lattices.

	1 pt spaces	2 pt spaces	3 pt spaces	4 pt spaces
$\lvert\mathbf{A}\rvert = 2$	1			
$\lvert\mathbf{A}\rvert = 3$	1	2		
$\lvert\mathbf{A}\rvert = 4$	3	6	5	
$\lvert\mathbf{A}\rvert = 5$	14	28	30	16

This is not a lot of data, but there are a few suggestive patterns.

I will call attention to one in particular.

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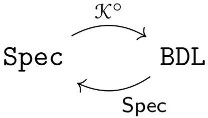
The numbers 1, 2, 5, 16 are the numbers of non-homeomorphic spectral topologies on a 1-, 2-, 3-, and 4-element set, respectively.

Conjecture

For every spectral space Y on $n-1$ elements, there is a BCK-algebra $\mathbf{A}$ of order n such that $\mathrm{X}(\mathbf{A}) \simeq Y$.



It is known that if $\mathrm{X}(\mathbf{A})$ is a Noetherian space, then $\mathcal{K}^{\circ}\mathrm{X}(\mathbf{A}) \cong \mathrm{Id}(\mathbf{A})$.

Conjecture

Every finite bounded distributive lattice is the ideal lattice of some finite BCK-algebra.

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