A Characterization of Finitely Based Abelian Mal'cev Varieties

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Outline

1. the Finite Basis Problem
2. Abelian Mal'cev Varieties
3. future directions

Finite Basis Problem

Algebras and Varieties

An *algebra* **A** is a set with operations.

Example

- the group structure $\langle \mathbb{Z}, +, -, 0 \rangle$
- a lattice $\langle L, \vee, \wedge \rangle$

The set of operations the algebra's *signature*.

A *variety* $\mathcal{V}$ is a class of algebras with the same signature that is defined by a set of equations Σ. Then Σ is a *basis* for $\mathcal{V}$.

Example

The variety of groups is the variety with signature $\langle +, -, 0 \rangle$ satisfying the following

G1 $(x + y) + z \approx x + (y + z)$

G2 $x + 0 \approx 0 + x \approx x$

G3 $x + (-x) \approx (-x) + x \approx 0$

Varieties generated by algebras

If **A** is an algebra, then $\mathcal{V}(\mathbf{A})$ is the variety of algebras satisfying all identities that **A** satisfies.

Equivalently, $\mathcal{V}(\mathbf{A})$ is the class of homomorphic images of subalgebras of direct powers of **A**.

Example

$\mathcal{V}(\langle \mathbb{Z}, +, -, 0 \rangle) =$ the variety of abelian groups

Finitely based varieties and algebras

A variety is *finitely based* if its basis of equations Σ is finite in size.

Example

- The variety of groups is finitely based.
- The variety of lattices is finitely based.

An algebra $\mathbf{A}$ is *finitely based* if $\mathcal{V}(\mathbf{A})$ is finitely based.

Example

The group $\langle \mathbb{Z}, +, -, 0 \rangle$ is finitely based.

The Finite Basis Problem

In the 1960's, the following question was asked:

Question (A. Tarski)

Is there an algorithm that, given a finite algebra as input, can determine if it is finitely based?

R. McKenzie (1996) provided a negative answer.
The proof involved constructing for each Turing Machine T an algebra $\mathbf{A}(T)$ which was finitely based if and only if T halted.

Question

Can we characterize finitely based algebras in well-behaved classes?

Examples of Finite Basis Results

Example

- (Lyndon, 1954) There exist finite algebras that are not finitely based.
- (S. Oates and M. Powell, 1964) Every finite group is finitely based.
- (R. Freese and R. McKenzie, 1987) Every finite supernilpotent Mal'cev algebra is finitely based.

Abelian Mal'cev Algebras

Mal'cev algebras

A variety $\mathcal{V}$ is Mal'cev if there exists a ternary term operation $m(x, y, z)$, called the *Mal'cev term*, such that

$$\mathcal{V} \models m(x, x, y) \approx y \approx m(y, x, x).$$

Example

Any expansion of additive groups has Mal'cev term $m(x, y, z) = x - y + z$.

Abelian Algebras

An algebra is *abelian* if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$, where $0_{\mathbf{A}}, 1_{\mathbf{A}}$ are the total and trivial congruences, respectively. A variety is abelian if its algebras are all abelian.

Theorem (H.P. Gumm, 1980)

A variety with Mal'cev term $m(x, y, z)$ is abelian if and only if for every basic operation f we have

$$\mathcal{V} \models m(f(\overline{x}), f(\overline{y}), f(\overline{z})) \approx f(m(x_1, y_1, z_1), \ldots, m(x_k, y_k, z_k)),$$

where f is assumed to have arity k and $\overline{x}$ is a k-tuple.

Example

$\langle \mathbb{Z}, x - y + z, 3x + 4y + 3 \rangle$ is an abelian algebra

Free Algebras

For a variety $\mathcal{V}$ and variables $x_1, \ldots, x_n$, we let $\mathbf{F}_{\mathcal{V}}(x_1, \ldots, x_n)$ denote the *free algebra over* $x_1, \ldots, x_n$.
A term $t(x_1, \ldots, x_n)$ is *idempotent* in a variety $\mathcal{V}$ if

$$\mathcal{V} \models t(z, \ldots, z) \approx z.$$

We let $F_{\mathcal{V}}^{id}(x_1, \ldots, x_n) = \{t \in F_{\mathcal{V}}(x_1, \ldots, x_n) \mid t(z, \ldots, z) = z\}$

Module Structure

Theorem (c.f. R. Freese, R. McKenzie, 1987)

Let $\mathcal{V}$ be an abelian variety with Mal'cev term m. For $s, t \in F_{\mathcal{V}}(x, z)$ define

$$\begin{aligned} s + t &:= m\big(s(x,z), z, t(x,z)\big), \\ s \cdot t &:= s(t(x,z), z) \\ -s &:= m(z, s(x,z), z). \end{aligned}$$

1. $\mathbf{R}_{\mathcal{V}} := \langle F_{\mathcal{V}}^{id}(x,z), +, -, \cdot \rangle$ is a ring with identity x and zero z.
2. $\mathbf{M}_{\mathcal{V}} := \langle F_{\mathcal{V}}(z), +, -, R_{\mathcal{V}} \rangle$ is an $\mathbf{R}_{\mathcal{V}}$-module with zero z.
3. $\langle F_{\mathcal{V}}(x,z), +, -, R_{\mathcal{V}} \rangle \cong R_{\mathcal{V}} \oplus \mathbf{M}_{\mathcal{V}}$ as $\mathbf{R}_{\mathcal{V}}$-modules

Characterization Theorem

Theorem (M. Muro, 2024)

Let $\mathcal{V}$ be an abelian Mal'cev variety. Then $\mathcal{V}$ is finitely based if and only if

1. $\mathcal{V}$ has finite signature,
2. the ring $\mathbf{R}_{\mathcal{V}}$ of binary idempotent terms is finitely presented,
3. and the $\mathbf{R}_{\mathcal{V}}$-module $\mathbf{M}_{\mathcal{V}}$ of unary terms is finitely presented.

Proof of Characterization Theorem

Lemma (Folklore)

Let $\mathcal{V}$ be an abelian variety with Mal'cev term m. Then $\mathcal{V}$ is equivalent to a variety $\mathcal{W}$ with operations induced by $F_{\mathcal{V}}(z) \cup F_{\mathcal{V}}^{id}(x,z) \cup \{m\}$. If $\mathcal{V}$ has finite signature, then $\mathcal{W}$ has signature $\{u_1, \ldots, u_\ell, r_1, \ldots, r_n, m\}$.

Proof.

Every term in $\mathcal{V}$ decomposes into a sum of unary and binary terms. For example, $\mathcal{V}$ models

$$\begin{aligned} f(x_1, x_2, z) &\approx f\left(m(x_1, z, z), m(z, z, x_2), m(z, z, z)\right) \\ &\approx m\left(f(x_1, z, z), f(z, z, z), f(z, x_2, z)\right) \\ &\approx f(x_1, z, z) - f(z, z, z) + f(z, x_2, z) \\ &\approx f(x_1, z, z) - f(z, z, z) + f(z, x_2, z) - f(z, z, z) + f(z, z, z). \end{aligned}$$

Note that $f(x_1, z, z) - f(z, z, z)$ is idempotent in x_1, z. Also, $f(z, x_2, z) - f(z, z, z)$ is idempotent in x_2, z.

□

Example of Equivalent Variety

Example

The variety $\mathcal{V}(\langle \mathbb{Z}, x - y + z, 3x + 6y + 3 \rangle)$ is equivalent to an abelian variety in signature $\langle x - y + z, 3x - 3z, 6y - 6z, 9z + 3 \rangle$.

Definition of $\mathcal{U}$

We define the auxilliary variety $\mathcal{U}$ to be the variety with signature $\{u_1, \ldots, u_\ell, r_1, \ldots, r_n, m\}$ satisfying the following

- $m(x, y, y) \approx x \approx m(y, y, x)$.
- $r_i(z, z) \approx z$ for all $1 \leq i \leq n$.
- $m\big(u_i(x), u_i(y), u_i(z)\big) \approx u_i\big(m(x, y, z)\big)$ for all $1 \leq i \leq \ell$.
- $m\big(r_i(x_1, x_2), r_i(y_1, y_2), r_i(z_1, z_2)\big) \approx$
  $r_i\big(m(x_1, y_1, z_1), m(x_2, y_2, z_2)\big)$ for all $1 \leq i \leq n$.
- $m\big(m(x_1, x_2, x_3), m(y_1, y_2, y_3), m(z_1, z_2, z_3)\big) \approx$
  $m\big(m(x_1, y_1, z_1), m(x_2, y_2, z_2), m(x_3, y_3, z_3)\big)$.

$\mathcal{U}$

Lemma

The variety $\mathcal{U}$ has the following properties

- The variety $\mathcal{U}$ is finitely based.
- The variety $\mathcal{U}$ is the largest abelian Mal'cev variety in the signature $\{u_1, \ldots, u_\ell, r_1, \ldots, r_n, m\}$.
- The ring $\mathbf{R}_{\mathcal{U}}$ is free over $r_1, \ldots, r_n$.
- The $\mathbf{R}_{\mathcal{U}}$-module $\mathbf{M}_{\mathcal{U}}$ is free over $u_1, \ldots, u_\ell$.
- The variety $\mathcal{W}$ is a subvariety of $\mathcal{U}$.

Morphisms

Define $\phi : \mathbf{F}_{\mathcal{U}}(x_1, \ldots) \to \mathbf{F}_{\mathcal{W}}(x_1, \ldots),\ t^{\mathcal{U}} \mapsto t^{\mathcal{W}}$.
By Birkhoff's Theorem, $\mathcal{W}$ is finitely based relative to $\mathcal{U}$ if and only if $\ker \phi$ is finitely generated as a fully invariant congruence of $\mathbf{F}_{\mathcal{U}}(x_1, \ldots)$.

Lemma (M. Muro, 2024)

Every fully invariant congruence of $\mathbf{F}_{\mathcal{U}}(x_1, \ldots)$ uniquely determines a pair of an ideal of $\mathbf{R}_{\mathcal{U}}$ and a submodule of $\mathbf{M}_{\mathcal{U}}$.
Conversely, every pair of an ideal of $\mathbf{R}_{\mathcal{U}}$ and a submodule of $\mathbf{M}_{\mathcal{U}}$ determines a fully invariant congruence of $\mathbf{F}_{\mathcal{U}}(x_1, \ldots)$.
Furthermore, the finitely generated fully invariant congruences of $\mathbf{F}_{\mathcal{U}}(x_1, \ldots)$ are in bijection with the finitely generated ideals and submodules.

We have all the parts to prove that $\mathcal{W}$ is finitely based if and only if $\mathbf{R}_{\mathcal{W}}$ and $\mathbf{M}_{\mathcal{W}}$ are finitely presented.
Let $\phi\colon \mathbf{F}_{\mathcal{U}}(x,\ldots) \to \mathbf{F}_{\mathcal{W}}(x,\ldots),\ t^{\mathcal{U}} \mapsto t^{\mathcal{W}}$ and let $\theta = \ker\phi$.

- If $\mathcal{W}$ is finitely based, then θ is finitely generated as a fully invariant congruence of $\mathbf{F}_{\mathcal{U}}(x_1,\ldots)$.
- This means that θ uniquely determines a finitely generated ideal I of $\mathbf{R}_{\mathcal{U}}$ and finitely generated submodule N of $\mathbf{M}_{\mathcal{U}}$.
- We have $\mathbf{R}_{\mathcal{W}} \cong \mathbf{R}_{\mathcal{U}}/I$ and $\mathbf{M}_{\mathcal{W}} \cong \mathbf{M}_{\mathcal{U}}/N$.
- Since $\mathbf{R}_{\mathcal{U}}$ and $\mathbf{M}_{\mathcal{U}}$ are free with finitely many generators, that means $\mathbf{R}_{\mathcal{W}}$ and $\mathbf{M}_{\mathcal{W}}$ are finitely presented.

The backwards direction is similar.

Future Directions

Relative Finite Basis

The natural next step is to work with nilpotent Mal'cev algebras. Given a Mal'cev variety $\mathcal{V}$ of finite type, is the subvariety $\mathcal{V}_n$ of n-nilpotent algebras in $\mathcal{V}$ finitely based relative to $\mathcal{V}$?

- We know this is true for $n = 1$.
- (R. Freese, R. McKenzie, 1987) We also know this is true when $\mathbf{F}_{\mathcal{V}}(x, z)$ is finite.

Non-abelian group structure

Recall that there was a addition on elements of the free algebra defined as

$$s + t := m(s, z, t)$$

- This defines a loop on any free algebra in a nilpotent $\mathcal{V}$.

References

[M. Muro, 2024] Characterizing Finitely Based Abelian Mal'cev Algebras - https://arxiv.org/abs/2411.17004