Characterizing some classical rings via superstability

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Background

- Abstract elementary classes were introduced by Shelah in the 70's.

Shelah's categoricity conjecture (1976)

Assume $\mathbf{K}$ is an AEC.
If $\mathbf{K}$ is categorical in *some* cardinal greater than or equal to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$, then it is categorical in *all* cardinals greater than or equal to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.

- Superstability.
- Abelian groups and modules

Outline

1. **Basic notions**
2. Limit models and stability
3. Superstability
4. Characterizing rings via superstability
5. Summary and future work

Basic notions: Module theory

R is an associative ring with unity.

R-modules (intuition)

- An R-module is a "vector space" over the ring R.
- $\mathbb{R}$-modules are vectors spaces over $\mathbb{R}$.
- $\mathbb{Z}$-modules are abelian groups.

Language

Given a ring R, $L_R = \{0, +, -\} \cup \{r\cdot : r \in R\}$ is the language of R-modules.

Basic notions: Module theory

Pure submodule: 1^{st} Definition (Prüfer)

$M \leq_p N$ if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism.

Pure submodule: 2^{nd} Definition

$M \leq_p N$ if and only if for every $\bar{a} \in M$ and ϕ an existentially quantified system of linear equations, if $N \vDash \phi[\bar{a}]$ then $M \vDash \phi[\bar{a}]$.

Pure subgroup

$M \leq_p N$ if and only if $kN \cap M = kM$ for every $k \in \mathbb{N}$.

Examples

$M \leq_p M \oplus N$, $t(G) \leq_p G$, $\mathbb{Z} \nleq_p \mathbb{Q}$.

Basic notions: Abstract elementary class (Shelah)

An *abstract elementary class of modules* (AEC) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where $K \subseteq R$-Mod and $\leq_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.

Key axioms

1. $\mathbf{K}$ is closed under isomorphisms.
2. If $M \leq_{\mathbf{K}} N$, then M is a submodule of N.
3. Tarski-Vaught axioms: Closed under unions of increasing chains.
4. Löwenheim-Skolem-Tarski axiom: For $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq M_0$ and M_0 is *small*.

In this talk small means that $\|M_0\| \leq |R| + \aleph_0 + |A|$.

Basic notions: Abstract elementary class

- $(Ab, \leq)$ and $(Ab, \leq_p)$.
- $(\text{Tor}, \leq)$ and $(\text{Tor}, \leq_p)$.
- $(\text{torsion-free}, \leq)$ $(\text{torsion-free}, \leq_p)$.
- $(RTF, \leq_p)$.
- $(\aleph_1\text{-free}, \leq_p)$.
- $(R\text{-Mod}, \leq)$ and $(R\text{-Mod}, \leq_p)$.
- $(\mathfrak{s}\text{-Tor}, \leq_p)$
- $(R\text{-Flat}, \leq_p)$.
- $(R\text{-Absp}, \leq_p)$.
- $(R\text{-l-inj}, \leq_p)$ and $(R\text{-l-pi}, \leq_p)$.

Basic notions: Some properties

$f : M \to N$ is a **K**-embedding if $f : M \cong f[M] \leq_{\mathbf{K}} N$.

Amalgamation property (AP)

Every $M \leq_{\mathbf{K}} N_1, N_2$ can be completed to a commutative square in **K**.

$$\begin{tikzcd} N_1 \arrow[r, "f"] & N' \\ M \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & N_2 \arrow[u, "g"'] \end{tikzcd}$$

Examples

- (Kucera-M.) AP: R-Mod with pure embeddings, Absolutely pure R-modules with pure embeddings.
- (Shelah) No AP: $\aleph_1$-free abelian groups with pure embeddings.
- (?) AP?: Finitely Butler groups with pure embeddings.

Basic notions: Some properties

Some properties

1. **K** has JEP: if every $M, N \in K$ can be **K**-embedded into a model in **K**.
2. **K** has NMM: if every $M \in K$ can be properly extended in **K**.

Examples

All the examples we introduced have JEP and NMM as they are closed under direct sums.

Hypothesis

K has AP, JEP and NMM.

Outline

Limit models and stability: Basic notions

Universal extension (Kolman-Shelah)

M is *universal over* N if and only if M and N have the same size, $N \leq_{\mathbf{K}} M$ and for any $N^* \in \mathbf{K}$ of the size of M such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\longrightarrow} M$.

Limit model (Kolman-Shelah)

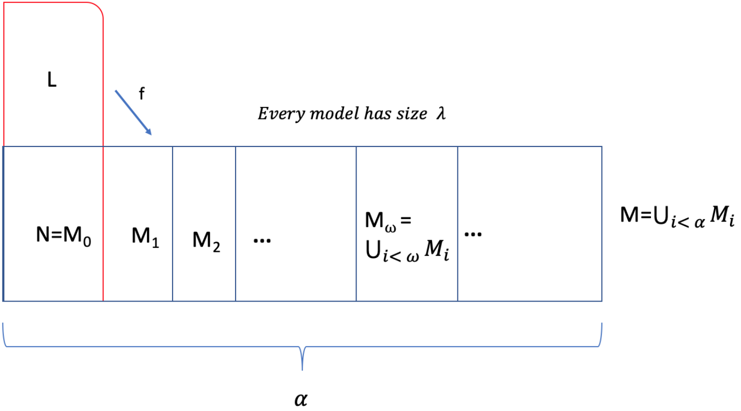
Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 := N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

M is a λ-limit model if there is $\alpha < \lambda^+$ limit ordinal and $N \in K_\lambda$ such that M is a (λ, α)-limit model over N.

Limit models and stability: Basic notions

Let $\alpha < \lambda^+$ limit ordinal. M is a (λ, α)-limit model over N.



(M.) Limit models in (Ab, $\leq$)

If M is a (λ, α)-limit model in (Ab, $\leq$), then $M \cong \mathbb{Q}^{(\lambda)} \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$.

Limit models and stability: Stability

Stable (Shelah)

- $\mathbf{K}$ is λ-stable if $\mathbf{K}$ has a λ-limit model.
- $\mathbf{K}$ is stable if there is a λ such that $\mathbf{K}$ is λ-stable.

This is not the standard definition of stability for AECs but its equivalent to it by work of Shelah, Grossberg and Van Dieren.

T is a complete first-order theory: $(Mod(T), \preceq)$

$(Mod(T), \preceq)$ is λ-stable if and only if T is λ-stable (as a first-order theory).

Theorem (Fisher-Bauer 70s)

If T is a complete first-order theory of modules, then $(Mod(T), \leq_p)$ is stable.

Question

Let R be an associative ring with unity.
If $(K, \leq_p)$ be an AEC of modules, is $(K, \leq_p)$ stable? Is this true if $R = \mathbb{Z}$?
Under what conditions on R is this true?

- Characterize Galois-types using first-order pp-types.
- Use a non-forking independence relation which is a notion similar to linear independence.

Limit models and stability: Stability for modules

Examples

- (Kucera-M.) (R-Mod, $\leq_p$).
- (Lieberman-Rosický-Vasey) (R-Flat, $\leq_p$).
- (M.) (R-Absp, $\leq_p$).
- (M.) (R-l-inj, $\leq_p$) and (R-l-pi, $\leq_p$).

Stability can be use to answer algebraic questions ...

Outline

Superstability: Definition

Key issue

Given M, N λ-limit models, are M and N isomorphic?

Fact (Shelah)

If M is a (λ, α)-limit model, N is a (λ, β)-limit model and $cf(\alpha) = cf(\beta)$, then M is isomorphic to N.

Uniqueness of limit models

$\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has a λ-limit model and if given M, N λ-limit models, M and N are isomorphic.

Superstability: Definition

Superstability (Shelah, Grossberg, Vasey)

K is *superstable* if and only if **K** has uniqueness of limit models in a tail of cardinals, i.e., there is a μ such that for every $\lambda \geq \mu$ **K** has uniqueness of limit models of cardinality λ.

If **K** is superstable, then **K** is stable.

T is a complete first-order theory: $(Mod(T), \preceq)$

$(Mod(T), \preceq)$ is superstable if and only if T is superstable.

Main question

If **K** is an AEC of modules, under what conditions is **K** superstable? Is there an algebraic reason why this happens?

Superstability: Assuming Hypothesis 1

Hypothesis 1

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under pure submodules.
3. K is closed under pure-injective envelopes, i.e., if $M \in K$, then $PE(M) \in K$.

Examples

- R-modules.
- Absolutely pure modules: For all N, $M \subseteq_R N$ implies $M \leq_p N$.

Superstability: Assuming Hypothesis 1

Pure-injective

M is pure-injective if for every N with $M \leq_p N$ we have that M is a direct summand of N, i.e., there is M' such that $N = M \oplus M'$.

Pure-injective envelope (Ziegler)

The pure-injective envelope of M, denoted by $PE(M)$, is a pure-injective module with $M \leq_p PE(M)$ and it is minimum with respect to these properties.

Long limit models (M.)

If M is a (λ, α)-limit model with $cf(\alpha) \geq (|R| + \aleph_0)^+$, then M is pure-injective.

Where $(|R| + \aleph_0)^+$ is the cardinal following $|R| + \aleph_0$

Superstability: Assuming Hypothesis 1

Shortest limit model (M.)

If M is a (λ, ω)-limit model and N is a $(\lambda, (|R| + \aleph_0)^+)$-limit model, then $M \cong N^{(\aleph_0)}$.

Question

Let M be a (λ, α)-limit model with $\omega < cf(\alpha) < (|R| + \aleph_0)^+$. What can we say about M?

Lemma (M.)

If M and N are limit models, then M and N are elementary equivalent,i.e., they look the same from the perspective of first-order logic.

Superstability: Assuming Hypothesis 1

Σ-pure-injective

M is Σ-pure-injective if and only if $M^{(\aleph_0)}$ is pure-injective

If a module is Σ-pure-injective then it is pure-injective.

Why are Σ-pure-injective modules important to us? (Gruson-Jensen, Zimmerman, Garavaglia)

- They are closed under elementary equivalence.
- They are closed under pure submodules.

Superstability: Assuming Hypothesis 1

Lemma (M.)

If there exists $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}$ has uniqueness of limit models of cardinality λ, then every limit model is Σ-pure-injective.

Proof sketch:

- N be a $(\lambda, (|R| + \aleph_0)^+)$-limit model.
- $N^{(\aleph_0)}$ is the (λ, ω)-limit model.
- N is isomorphic to $N^{(\aleph_0)}$ by uniqueness of limit models.
- $N^{(\aleph_0)}$ is pure-injective because N is pure-injective.
- N is Σ-pure-injective.
- Every limit model is Σ-pure-injective because it is elementarily equivalent to N.

Superstability: Assuming Hypothesis 1

Theorem (M.)

The following are equivalent.

1. $\mathbf{K}$ is superstable.
2. Every limit model in $\mathbf{K}$ is Σ-pure-injective.
3. Every model in $\mathbf{K}$ is pure-injective.

Outline

Characterizing rings via superstability: Noetherian rings

Noetherian rings (1921): Every absolutely pure module is injective.

Characterizing noetherian rings (M.)

Let R be an associative ring with unity. The following are equivalent.

1. R is left noetherian.
2. The class of absolutely pure left R-modules with pure embeddings is superstable
3. The class of left R-modules with embeddings is superstable.

(1) $\leftrightarrow$ (2) follows directly from the theorem we obtained under hypothesis 1.

Characterizing rings via superstability: Pure-semisimple rings

Pure-semisimple rings (1977): Every R-module is pure-injective.

Characterizing pure-semisimple rings (M.)

Let R be an associative ring with unity. The following are equivalent.

1. R is left pure-semisimple
2. The class of left R-modules with pure embeddings is superstable

Follows directly from the theorem we obtained under hypothesis 1.

Abelian groups (M.)

$(Ab, \leq)$ is superstable, but $(Ab, \leq_p)$ is not superstable.

Characterizing rings via superstability: Perfect rings

Perfect rings (1960): Every flat module is a projective module.

Characterizing perfect rings (M.)

Let R be an associative ring with unity. The following are equivalent.

1. R is left perfect.
2. The class of flat left R-modules with pure embeddings is superstable.

How does this case compare to previous cases?

- Flat modules are NOT closed under pure-injective envelopes so we need to work with cotorsion modules.
- (Σ-) Cotorsion modules are NOT as nice a (Σ-)pure-injective modules.

Characterizing rings via superstability: $\mathfrak{s}$-torsion modules

(Martsinkovsky-Russell) $\mathfrak{s}$-torsion modules are a generalization of torsion abelian groups.

Characterizing superstability (M.)

Let R be an associative ring with unity such that for every N, $\mathfrak{s}(N) \leq_p N$. The following are equivalent.

1. The class of $\mathfrak{s}$-torsion R-modules with pure embeddings is superstable.
2. Every limit model is Σ-$\mathbf{K}^{\mathfrak{s}\text{-Tor}}$-pure-injective.
3. Every $\mathfrak{s}$-torsion modules is $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$-pure-injective.

Characterizing rings via superstability: $\mathfrak{s}$-torsion modules

How does this case compare to previous cases?

- $\mathfrak{s}$-torsion modules are NOT closed under pure-injective envelopes or cotorsion envelopes so we need to work with $\mathbf{K}^{\mathfrak{s}\text{-}\mathrm{Tor}}$-pure-injectives.
- Nobody had studied $\mathbf{K}^{\mathfrak{s}\text{-}\mathrm{Tor}}$-pure-injective modules, so I had to develop the algebraic theory.
- It is open whether superstability for $\mathfrak{s}$-torsion modules characterizes a classical class of rings.

Abelian groups (M.)

$(\mathrm{Tor}, \leq_p)$ is not superstable.

Outline

Summary and future work

Summary

- There are many natural AECs of modules.
- Characterizing limit models algebraically is the key to understand superstability.
- Superstability is a natural algebraic property.

Future work

- Are all AECs of modules with pure embeddings stable?
- Characterize the limit models algebraically for other AECs of modules.
- Do the algebraic characterizations of superstability extend to other algebraic settings? Ongoing project with Rosický.
- Use AECs of modules to answer algebraic questions.

References

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Thank you!