

Finite algebras whose equational probability spectrum contains a big gap

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- ④ Miklós Abért showed in 2006 that if G is nonsolvable, but every quotient of G is solvable, then the set of all probabilities of equations is dense in $[0, 1]$.

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Definitions

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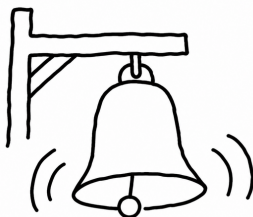
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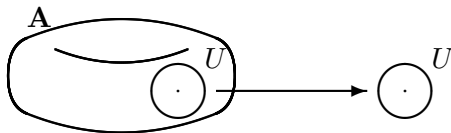
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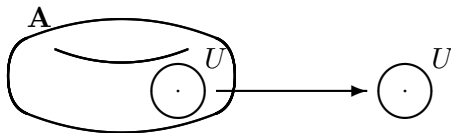
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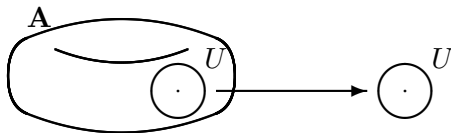
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- 3 If $\mathbf{A}$ is minimal and abelian, then $\mathbf{A}$ is polynomially equivalent to a finite G -set or a finite vector space.

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If $\alpha_i = \beta_i$ holds for all i , then $s \approx t$ has the same set of solutions as $c \approx d$. If $c = d$, then $\text{Prob}(s \approx t | \mathbf{A}) = 1$, while if $c \neq d$, then $\text{Prob}(s \approx t | \mathbf{A}) = 0$.

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If $\mathbf{A}$ satisfies the Big Gap hypothesis, then the group must act on $\mathbf{A}$ in such a way that every nonidentity group element has at most one fixed point.

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(Otherwise, in the 5th item $\frac{1}{|A|} < \frac{m}{|A|} < 1$.)

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- ④ same as previous case, but 0 is the image of a constant term.

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Additional comments

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