Locally finite Schreier varieties

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$$P_{\mathbf{A}} \neq \emptyset \Longleftrightarrow C_{\mathbf{A}} \neq \emptyset \Longleftrightarrow C_{\mathbf{A}} = P_{\mathbf{A}}.$$

That is, the range of any pseudoconstant is the interpretation of a constant in a Schreier variety.

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We have seen why any finite algebra in a locally finite Schreier variety must be $0, 1$-minimal. What is involved in proving the converse? Why is it the case that if every finite algebra in a variety is $0, 1$-minimal, then the variety is Schreier?

Assume that every finite algebra in $\mathcal{V}$ is $0, 1$-minimal. This means that each algebra in $\mathcal{V}$ satisfies $|\mathbf{A}| \leq 2$, $\mathbf{A}$ is polynomially equivalent to a G-set, or $\mathbf{A}$ is polynomially equivalent to a vector space.

Stage 1. Show that only one of these cases occurs in $\mathcal{V}$. I.e.,

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Stage 2. Examine the structure of the free algebras in varieties of these types and show that subalgebras of free algebras are free.

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