Geometry of involutions in ranked groups

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$SO_3(\mathbb{R})$ vs $PGL_2(\mathbb{C})$

An inner-geometric dividing line

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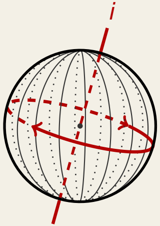

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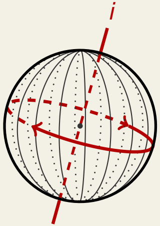

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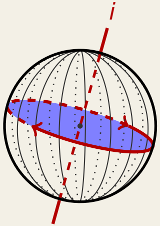

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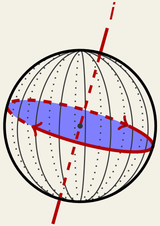

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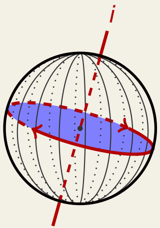

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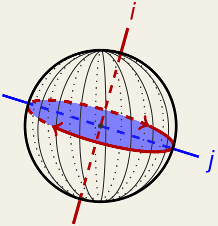

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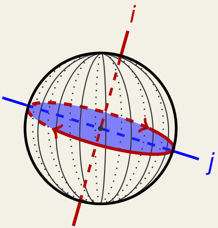

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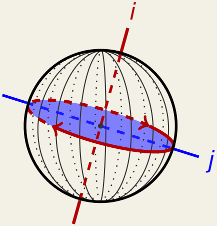

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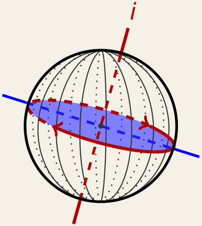

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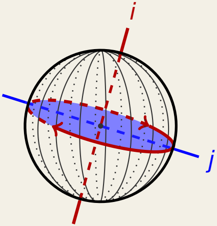

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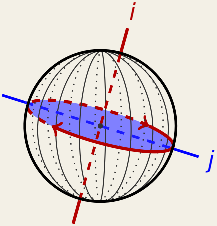

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- The geometries can more-or-less be reconstructed from the properties of the subgroup C. *This is the main point we want to explore.*

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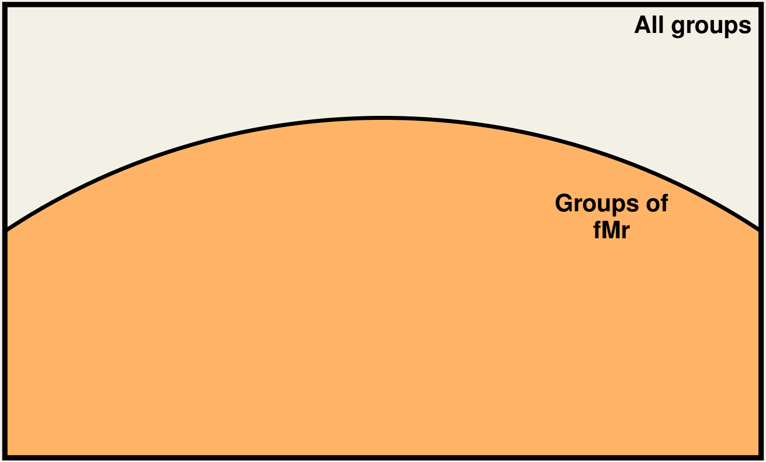
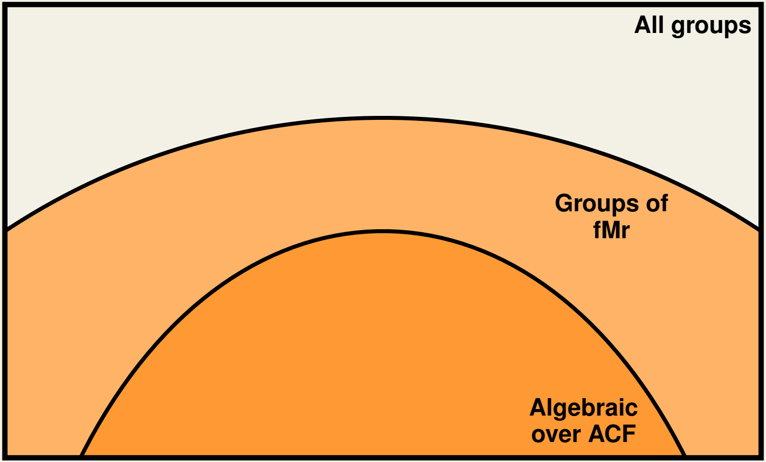
$SO_3(\mathbb{R})$ vs $PGL_2(\mathbb{C})$

A model-theoretic dividing line


All groups
Groups of fMr

Groups of finite Morley rank (fMr)



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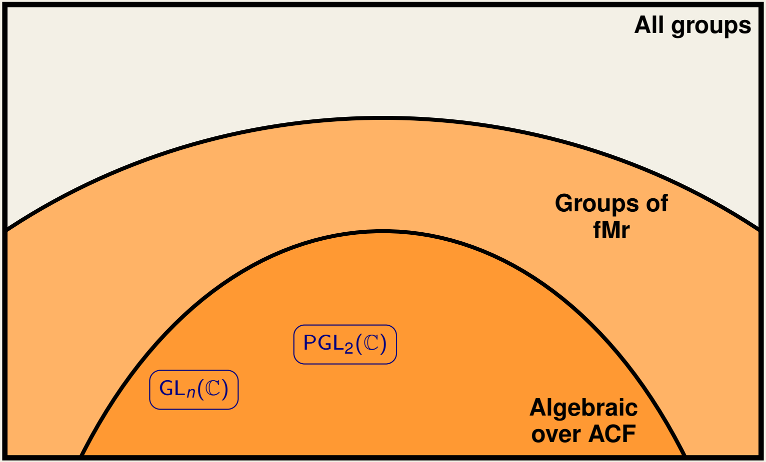

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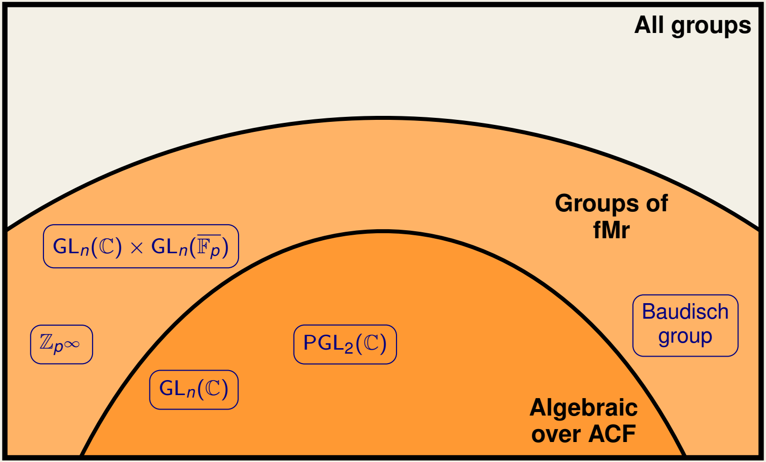

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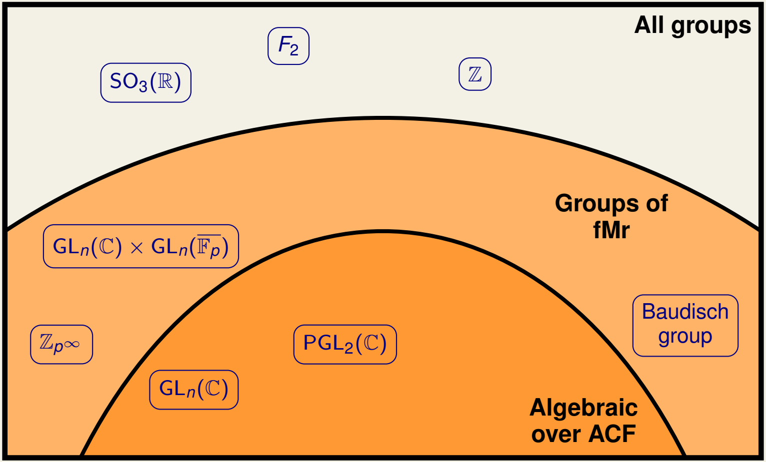

The broader context

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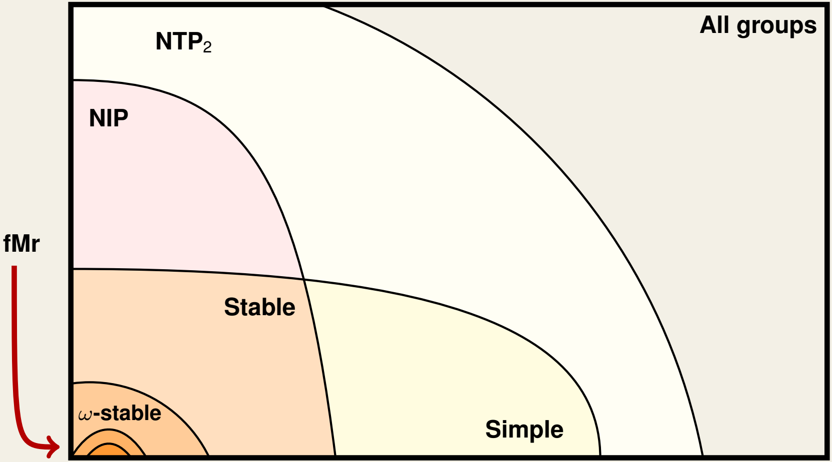

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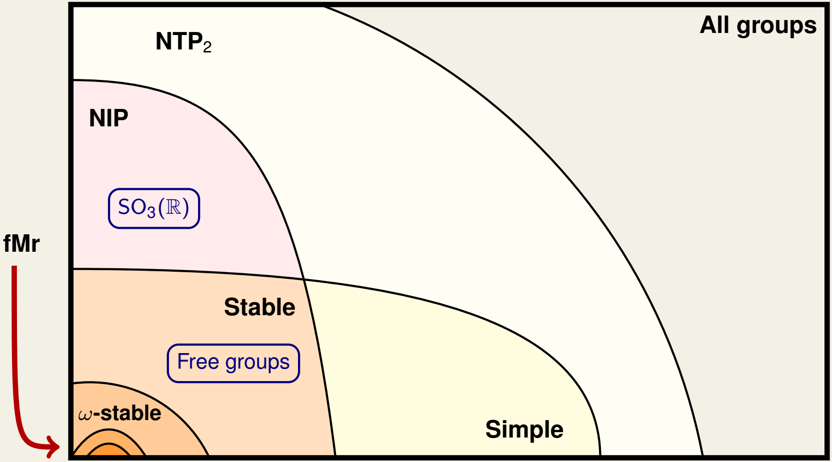
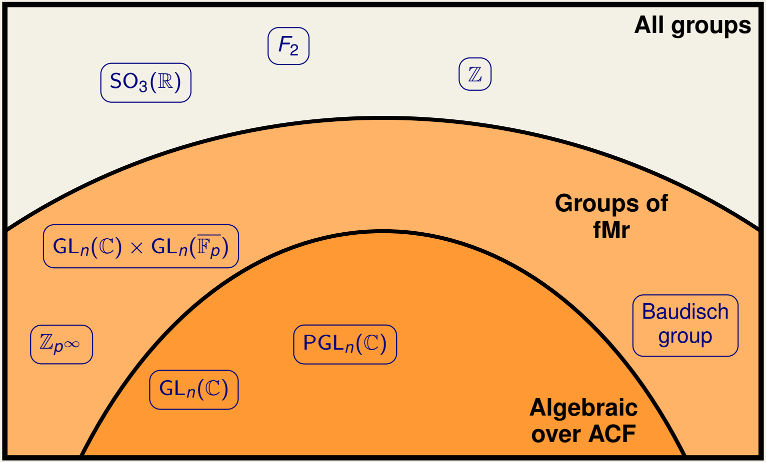


Algebraicity Conjecture



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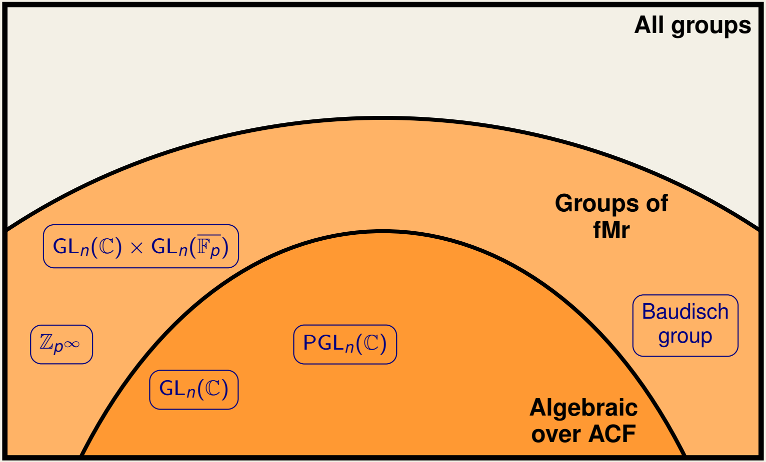

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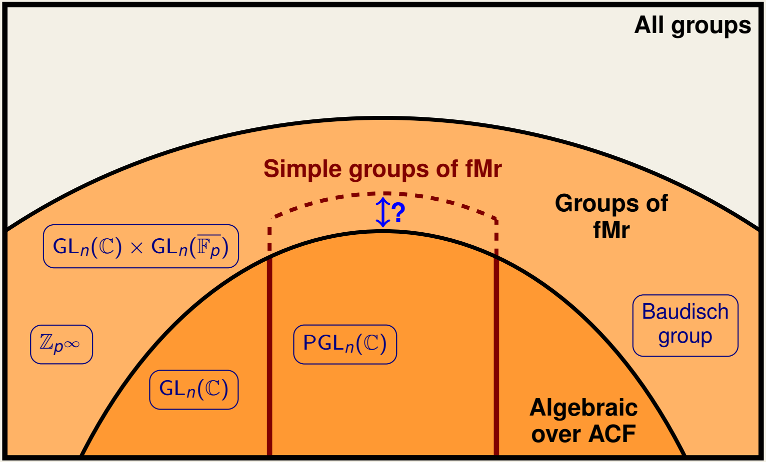

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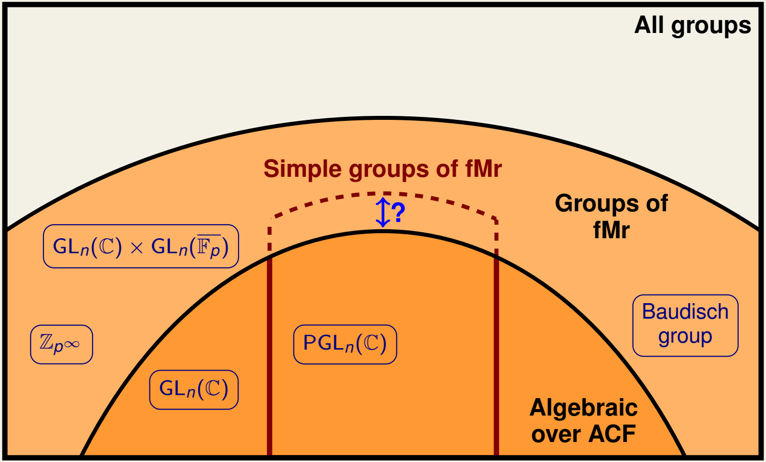

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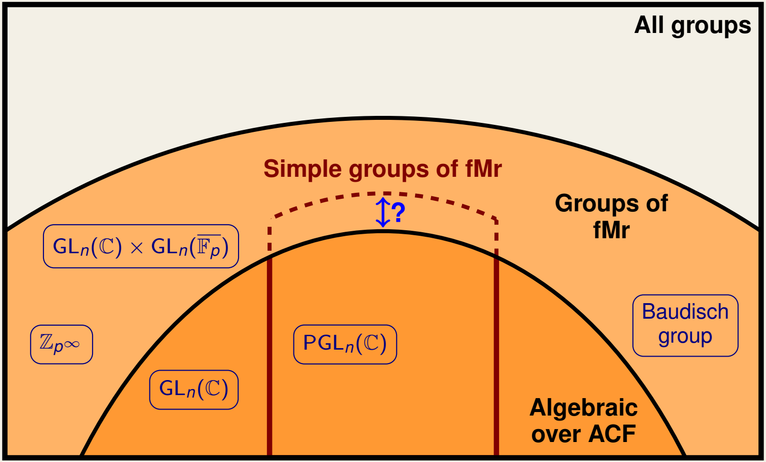

Algebraicity Conjecture: the gap, $\updownarrow$, does not exist.

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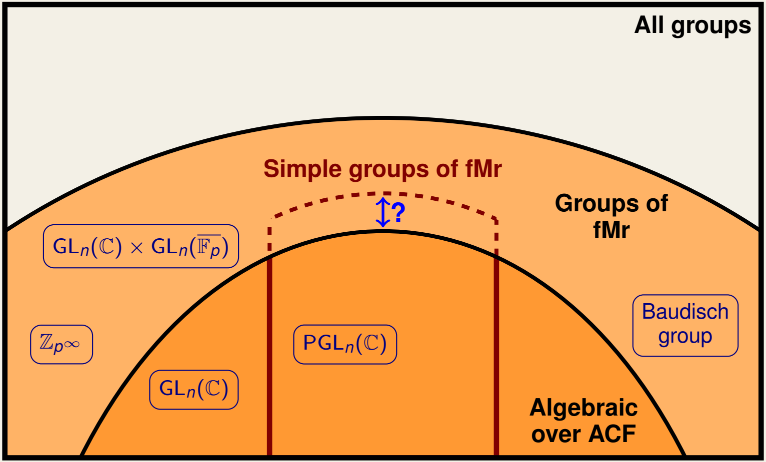

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- In general, the less 2-torsion a group has, the harder it becomes to analyze by "standard/generic" methods.

Geometry of involutions in quasi-Frobenius groups

The setting

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In fact, 1. $\implies$ (2. $\iff$ 2.' conjugates of C generically cover G.)

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Let G be a connected group of fMr with $0 < m_2(G) < \infty$. Suppose G is quasi-Frobenius with respect to $C < G$, i.e.

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Towards the main result

Both $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{PGL}_2(\mathbb{C})$ satisfy all of the group-theoretic conditions in our Global Hypotheses (as well as $[N_G(C) : C] = 2$). However, there is a difference:

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Towards the main result

Both $SO_3(\mathbb{R})$ and $PGL_2(\mathbb{C})$ satisfy all of the group-theoretic conditions in our Global Hypotheses (as well as $[N_G(C) : C] = 2$). However, there is a difference:

- the conjugates of C genuinely cover $SO_3(\mathbb{R})$
- the conjugates of C generically but not genuinely cover $PGL_2(\mathbb{C})$

We reframe this dividing line using the following notion.

Definition

An element is strongly real (SR) if it is inverted by some involution.

Example

All elements of $SO_3(\mathbb{R})$ are strongly real. The same is true in $PGL_2(\mathbb{C})$.

- the conjugates of C contain all SR elements of $SO_3(\mathbb{R})$
- the conjugates of C do not contain all SR elements of $PGL_2(\mathbb{C})$

The main result and consequences

The Geometric Theorem (Deloro-W)

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Corollary (Borovik-Burdges)

If $G \leq \mathrm{GL}_n(K)$ is simple, definable, but not Zariski closed for K of fMr in characteristic 0, then G has no involutions.

Proof sketch

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That it is genuine hinges on the conjugates of C genuinely covering G.

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Final thoughts

Reflections

The Geometric Theorem (Deloro-W)

Let G be a connected group of fMr with $0 < m_2(G) < \infty$*. Suppose G is quasi-Frobenius with respect to* $C < G$*. Then the conjugates of C do* <u>*not*</u> *contain all* SR *elements of G.*

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Recognition results

Theorem (Zamour)

Let G be a connected group of fMr with $0 < m_2(G) < \infty$. Suppose G is a nonsolvable quasi-Frobenius group with respect to $C < G$. If C is solvable and $[N_G(C) : C] = 2$, then either C is a maximal connected solvable of G or $G \cong \mathrm{PGL}_2(K)$.

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- So what might this be working towards…

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Moreover, in a suitable 'dimensional' setting (generalizing the fMr and o-minimal contexts), one still finds nice recognition of G, including now $\mathrm{PGL}_2(K)$, $\mathrm{SO}_3(\mathcal{R})$, and other related groups.

Thank You