Infinite Combinatorics from Finite structures
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Steiner systems and quasi-groups

Steiner Systems

A Steiner system with parameters t, k, n written $t - S(n, k, 1)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We begin with $t = 2$ and allow infinite n.

Some History

For which n's does an $2 - S(n, k, 1)$ system exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

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Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939); Skolem ($6n + 1$, 1958)

Linear Spaces

Definition: linear space

The vocabulary contains a single ternary predicate R, interpreted as collinearity. A linear space satisfies

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

α is the iso type of $(\{a, b\}, \{c\})$ where $R(a, b, c)$.

Groupoids and quasi-groups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

The connection between Steiner systems and quasigroups

1. Every Steiner triple system is a quasigroup.
   I.E. R is the graph of $\circ$.
2. Every p^i-Steiner system *admits* a compatible quasigroup structure. [GW75]
3. The [BP21] strongly minimal p^i-Steiner systems are not quasigroups (unless $p^i = 3$) [BV24].
4. There are strongly minimal Steiner groups $(A, R, *)$, that induce q-Steiner systems for every prime power q [Bal23].

Constructing infinite block designs: Amalgamation Classes

Constructing generic models

$\leq$-amalgamation Classes

A $\leq$-amalgamation class $(\boldsymbol{L}_0^*, \leq)$ is a collection of finite structures for a vocabulary σ (which may have function and relation symbols) satisfying [BS96]:

1. $\leq$ is a partial order refining $\subseteq$.
2. $\leq$ satisfies joint embedding and amalgamation.
3. $A, B, C \in \boldsymbol{L}_0^*$, $A \leq B$, and $C \subseteq B$ then $A \cap C \leq C$.
4. $\boldsymbol{L}_0^*$ is countable

Theorem

For a $\leq$-amalgamation class, there is a countable structure M, the *generic model*, which is a union of members of $\boldsymbol{L}_0^*$, each member of $\boldsymbol{L}_0^*$ embeds in M, and M is $\leq$-homogeneous.

For Fraïssé, the language is finite relational, the class is closed under substructure, and $\leq$ is $\subseteq$.

Existentially closed 3-Steiner Systems

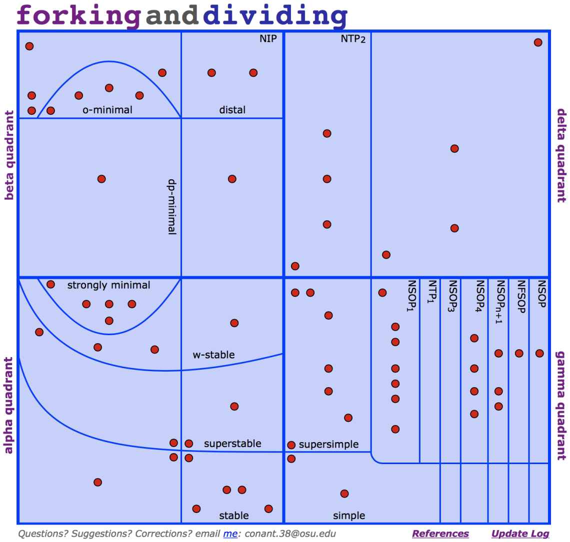
Barbina-Casanovas

[BC19] Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are each the graph of a Steiner quasigroup $(2-(n, 3, 1)$ system$)$.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T^*_{sq}.
2. T^*_{sq} has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and locally finite;
   4. T^*_{sq} has TP_2 and $NSOP_1$.

Key foundation: Every partial finite Steiner triple system can be embedded in a finite Steiner triple system. [Tre71]
See [Bal] for a 2009 survey of Hrushovski constructions.

Classification of first order theories



Gabe Conant's diagram

Omitting classes of Steiner quasigroups

Horsley- Webb

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures $(A, *)$ which are Steiner quasigroups that are F-free (omit a family $\boldsymbol{F}$ of finite nontrivial STS) and good (there exists an $A \in \boldsymbol{K}$ which neither extends nor embeds in any member of $\boldsymbol{F}$).

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus
2. $\tilde{\boldsymbol{K}}$ has a countable locally finite generic model.

On locally finite quasigroups their homogeneity is the model theorists ultrahomogeneity. Thus their construction gives $2^{\aleph_0}$ countable ($\aleph_0$ categorical) Steiner systems.

Question

Where do they fit on the map?

If $F = \emptyset$, this is T^*_{sq}. The others should be nearby.

Strongly Minimal Theories

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Theorem

If T is strongly minimal, algebraic closure defines a matroid/combinatorial geometry.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Usually this acl pre-geometry is not definable.

Constructing Strongly minimal Steiner systems

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive) $(\mathbb{Z}, S)$
2. vector space-like (lattice of subspaces modular) $(\mathbb{Q}, S)$
3. 'bi-interpretable' with an algebraically closed field (non-locally modular) $(\mathcal{C}, +, \times)$

Hrushovski disproved the conjecture by providing a method to construct strongly minimal sets that have flat geometries, admit no associative binary function, and more.

The flexibility of the Hrushovski construction

The ‘Hrushovski construction’ actually has 5 parameters:

Describing Hrushovski constructions

1. σ: vocabulary $\boldsymbol{L}_0^*$ is the collection all finite σ-structures.
2. $\boldsymbol{L}_0$: A $\forall\exists$ axiomatized subclass of $\boldsymbol{L}_0^*$
3. ϵ: A function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$ that induces a dimension on the definable subsets of the generic.
4. $\boldsymbol{L}_0 \subseteq \boldsymbol{L}_0^*$ defined using ϵ.
5. L_μ: the $A \in \boldsymbol{L}_0$ satisfying that the number of 0-primitive (B/C) are bounded by $\mu(B/C)$.

Choosing nice classes **U** of μ yields a collection of T_μ with similar properties.

For Hrushovski, the ‘standard’ **U** is $\mathcal{U} = \{\mu : \mu(C/B) \geq \delta(B)\}$.

Obtaining strong minimality

Primitive Extensions and Good Pairs

Let $A, B, C \in \mathbf{K}_0$.

(1) C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.

(2) C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

Bounding realization of good pairs

1. For any good pair (C/B), $\chi_M(B, C)$ is the maximal number of disjoint copies of C over B appearing in M.
2. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).

This guarantees strong minimality.

Strongly minimal linear spaces

Definition

A k-Steiner system is a $2-(n,k,1)$ block design (i.e. has n-elements, blocks (lines) with k elements, and 2 points determine a line).

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k.

The construction with $\mu(\alpha) = q - 2$ gives a q-Steiner system.

Hrushovki's basic construction vs Steiner

The relations are all taken as hypergraphs -define sets not sequences.

Example

1. σ has a single ternary relation R;
2. $\boldsymbol{L}_0$: All finite σ-structures
   finite linear spaces
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
   $\delta(A) = |A| - \sum_{\ell \in L(A)}(|\ell| - 2)$.
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
   Replace ϵ by δ.
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.
   with the exception: $\mu(\alpha) = q - 2$ gives line length q.

Here α is the primitive extension A/B where $B = \{a, b\}$ and $A = \{a, b, c\}$ with $R(a, b, c)$.

Strongly minimal Steiner Systems Exist

Theorem (Baldwin-Paolini)[BP21]

For each $k \geq 3$, there are an uncountable family T_μ for $\mu \in \mathcal{U}$, of strongly minimal $2-(\kappa, k, 1)$ Steiner-systems.

There is no infinite group definable in any T_μ.

Highly Transitive Block Designs

Motivation

1. Combinatorics:
   [Eva04] proves 'For all reasonable finite t, k and s we construct a $t-(\aleph_0, k, 1)$ design and a group of automorphisms which is transitive on blocks and has s orbits on points.'
   He suppresses the model theory and the paper appeared in the J. of Combinatorial design.
2. Model Theory:

Definition

[FM23] (Degree of nonminimality of a stationary type in a stable theory) Given a stationary type $p \in S(A)$, in a stable theory T with $\boldsymbol{U}(p) > 1$, the degree of nonminimality, nmdeg, of p is the minimal length n of a sequence of realizations of the type p, say $a_1, \ldots, a_n$ such that p has a nonalgebraic forking extension over $a_1, \ldots, a_n$.

It is known that such an n exists. The goal is to discover conditions on T to give uniform bounds for $\mathrm{nmdeg}(p)$.
Freitag discovered that 'high transitivity' is the key criteria.

Highly Transitive Block Designs

Definition

A $t-(v, k, \lambda)$ design is a set P of points, of cardinality v, together with a set $\boldsymbol{B}$ of blocks each of which is a k-element subset of P, and which has the property that any set of t points is a subset of exactly λ blocks. An automorphism of the design (P, B) is simply a permutation of P which sends blocks to blocks.

Theorem

For every $r, k < \omega$ there are $\aleph_1$-categorical theories $T_{r,k,\mu}$ whose models are $t-(\kappa, k, 1)$ block designs that have two orbits of s-sets for $s \leq t$.

(joint work with Freitag and Mutchnik) by varying the parameters of the Hrushovski construction.

Hrushovki's basic construction vs Highly Transitive

Requirements

For a theory $T_{n,k}$

1. σ has a single ternary relation R;
   σ has a single r-ary relation R and a unary P;
2. $\boldsymbol{L}_0$: All finite σ-structures
   finite r-space: $r-1$ pts determine a curve
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
   $\delta(A) = 2 \times |A| - |P(A)| - \sum_{\ell \in L(A)} (n_A(\ell)$
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
   Replace ϵ by δ. To get (almost) k transitivity: for $|A| \leq k$, forbid all B from $\boldsymbol{L}_0^*$ such that $B \supseteq A$ and $\delta(B/A) < 0$.
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.
   $\mu(\boldsymbol{\alpha}) = q - (r-1)$ gives line length q.

Highly Transitive Block Designs: result

Theorem

For every $r, k \geq r$, there is an almost strongly minimal theory $T_{r,k}$ with two sorts P and $\neg P$ such that for any $M \models T_{r,k}$ with $|M| = \kappa$

1. M is a τ-structure where τ has an r-ary relation R and a unary predicate P.
2. $M \subseteq \mathrm{acl}(P(M) \cup \boldsymbol{a})$ for a finite sequence $\boldsymbol{a}$;
3. M is the universe of an $(r-1)-(\kappa, \kappa, 1)$ design and the restriction to a strongly minimal subset P, is an $(r-1)-(\kappa, k, 1)$ design. There are two orbits on points: P and $\neg P$, the action of $\mathrm{aut(M)}$ is transitive on blocks.
4. Moreover for each $s \leq r$, the action of $\mathrm{aut(M)}$ (in P and in $\neg P$ is transitive on s element sets.
5. For $T_{r,k,\mu}$ the complete type over $\emptyset$, p, given by $\neg P(x)$ has $\mathrm{nmdeg}(p) = F_{ind}(p) = r - 1$.

Small Intersection Property

Largeness in permutation groups

Let M be a countable infinite structure, $G := \mathrm{autM}$.

- G is a topological (Polish) group under **pointwise convergence**: basic open sets are cosets of stabilizers of finite tuples over M

$$G_{m_1,\ldots,m_k} := \{g \in G \mathrm{st} g(m_i) = m_i \text{ for all } i \leq k\}.$$

- M has the **small index property (SIP)** if each $H \leq G$ of index $< 2^{\aleph_0}$ is open.
- G has **uncountable cofinality** if it is not a countable union of a chain of proper subgroups.
- G has the **Bergman property** if for each generating set $1 \in E = E^{-1}$ of G there exists $k \in \mathbb{N}$ such that $G = E^k$.
- G has **ample generics** if for each $n \in \mathbb{N}$ the conjugacy action of G on G^n has a comeager orbit (i.e. one containing the intersection of countably many dense open subsets of G^n).

Variants on SIP

$\mathrm{tp}(\boldsymbol{a}/B)$ is the collection p of formulas $\phi(\mathbf{x}; \mathbf{b})$ such that $\models \phi(\boldsymbol{a}/B)$. M realizes p if some $\boldsymbol{a} \in M$ satisfies p.
M is κ-saturated if every type of any $B \subset M$ with $|B| < \kappa$ is realized in M.
For uncountable saturated M, $\mathrm{aut(M)}$ has sip [LS93].
$|M| = \aleph_0$

1. M $\aleph_0$-saturated.
2. M is $\aleph_0$-categorical.

Each of 1) and 2) have some M with and some without sip.
We consider 1)

1. SIP Fails for the countable saturated model of algebraically closed fields and for $\overline{Q}$. [Las02]
2. True for the countable saturated model of the infinite rank ω-stable Hrushovski construction.

In general if $\mathrm{aut(M)}$ has sip, the group structure determines determines the Polish topology on autM.
For countable M

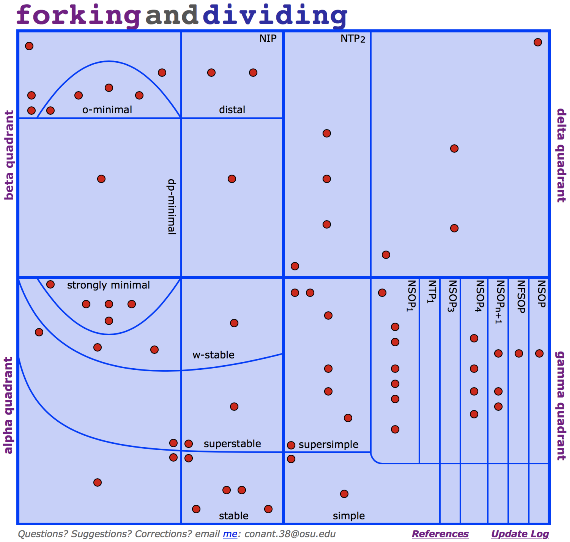
Theorem (Kechris, Rosendal 2007)

1. *Ample generics imply SIP.*
2. *For ω-categorical M, ample generics imply uncountable cofinality and the Bergman property for autM.*

Some countable examples

	SIP	uncountable cofinality	Bergman	ample generics
$(\mathbb{N}, =)$	Dixon Neumann Thomas '86	Macpherson Neumann '86	Bergman '06	Kechris Rosendal '07
random graph	Hodges Hodkinson Lascar Shelah '93	Hodges Hodkinson Lascar Shelah '93	Kechris Rosendal '07	Hrushovsky '92
$(\mathbb{Q}, \leq)$	Truss '89	Gourion '92	Droste Holland '05	no, Hodkinson
free group of rank ω	Bryant Evans '97	Bryant Evans '97	Tolstykh '07	Bryant Evans '97
Cantor space	Truss '87	Droste Göbel '05	Droste Göbel '05	Kwiatkowska '12

The model theoretic universe



Gabe Conant's diagram

Hrushovki's basic construction vs SIP for infinite rank

Example

1. σ has a finite relational language;
2. $\boldsymbol{L}_0$: All finite σ-structures
   SAME
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing R.
   Count each relation symbol
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$. SAME
5. **U** is those μ with $\mu(A/B) \geq \epsilon(B)$.
   $\mu(\boldsymbol{\alpha}) = q - 2$ gives line length q. OMIT

preprint of Ghadernezad shows

Theorem

The generic structure (ω-saturated) $(\boldsymbol{L}^*, \epsilon)$ structure has SIP

Questions on SIP

Question

Which if any of the following have sip?

1. *Strongly minimal Steiner Systems? [BP21]*
2. *Almost strongly minimal ($M = \mathrm{acl}(D(M))$ D strongly minimal)*
   1. *The asm projective plane [Bal94]*
   2. *The asm highly transitive structures above*
3. *Stable 'Hrushovki structures'*
   1. *Spencer-Shelah random graph [BS97]*
   2. *Hrushovski's strictly stable $\aleph_0$-categorical theory [Her95]*

Note that Lascar [Las92] proves an 'almost sip' for strong types.

Coordinatization by varieties of algebras

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Given a (near) field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on F by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

This block algebra is a Steiner quasigroup with cardinality q.

Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras V '(weakly) coordinatizes' a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatized

A collection of algebras V definably coordinatizes a class $\mathcal{S}$ of k-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Theorem

[] [GW75, GW80] For each q, the class of q-Steiner systems is (weakly) coordinatized by a $(2, q)$-variety V of block algebras

Can this coordinatization be definable in the strongly minimal (M, R)?

No nontrivial definable binary functions [BV24]

$\mathrm{dcl}^*(X) = \mathrm{dcl}(X) - \bigcup_{Y \subsetneq X} \mathrm{dcl}(Y)$.

Theorem

Let T_μ be a strongly minimal theory as in original construction or Steiner type. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B)\}$). Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

G_I If T_μ triples, i.e.

$$\mu \in \{\mu : \mu(A/B) \geq 3\}$$

then $\mathrm{dcl}^*(I) = \emptyset$, $\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$, and every definable function is essentially unary.

$G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$, $\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$ and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly binary functions.

Thus for any $\mu \in \mathcal{U}$, T_μ does not admit elimination of imaginaries and the algebraic closure geometry is not disintegrated.

Necessary Definitions

Definition

1. [Pad72] An (r, k) variety is one in which every r-generated algebra has cardinality k and is freely generated by every r-element subset.
2. Mikado Variety A variety V of binary, idempotent algebras, $(2, k)$ algebras is called Mikado.

Constructing a strongly minimal quasigroup

Definition: $\boldsymbol{K}^q$

1. Fix a prime power q and a Mikado variety V of quasigroups such that F_2, the free algebra in V on 2 generators has q elements.
2. σ has two ternary relations R, H.
3. Let $\boldsymbol{K}^q_V$ be the collection of finite A such that (A, R)- is a q-linear spaces A, with (ℓ, H) a copy of the free V algebra on two elements, H holds only between elements of a line.
4. Any collinear triple extends to a q-element clique. (A $\forall\exists$ sentence.)

Since V is axiomatized by 2-variable equations, if $A' \in \boldsymbol{K}^q_V$, $A' \restriction H$ is the graph of an algebra in V. In the generic model *each pair* is included in a q-element line; but not in the finite structures.

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