Isotropy Groups of Quasi-Equational Theories

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Introduction

- **Isotropy** is a (new) mathematical phenomenon with manifestations in category theory, (universal) algebra, and theoretical computer science.
- We will see that isotropy encodes a generalized notion of *conjugation* or *inner automorphism* for many prominent categories in mathematics.

Motivation

- Recall that an automorphism α of a group G is *inner* if there is an element $s \in G$ such that α is given by *conjugation* with s, i.e.

$$(g \in G) \qquad \alpha(g) = sgs^{-1}.$$

- It turns out that the inner automorphisms of a group can be characterized *without* mentioning conjugation or group elements at all!

Motivation

- To see this, observe first that if α is an inner automorphism of a group G (induced by $s \in G$), then for each group morphism $f : G \to H$ with domain G we can 'push forward' α to define an inner automorphism

$$\alpha_f : H \xrightarrow{\sim} H$$

by conjugation with $f(s) \in H$ (so that $\alpha_{\mathbf{id}_G} = \alpha$), and this family of automorphisms $(\alpha_f)_f$ is *coherent*, in the sense that it satisfies the following *naturality* property: if $f : G \to G'$ and $f' : G' \to G''$ are group homomorphisms, then the following diagram commutes:

$$\begin{array}{ccc} G' & \xrightarrow{\alpha_f} & G' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f'} \\ G'' & \xrightarrow[\alpha_{f' \circ f}]{} & G'' \end{array}$$

Bergman's Theorem

For a group G, let us call an *arbitrary* family of automorphisms

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=G}$$

with the above naturality property an *extended inner automorphism* of G.

Theorem (Bergman [1])

Let G be a group and $\alpha : G \xrightarrow{\sim} G$ an automorphism of G. Then α is an **inner** *automorphism of G iff there is an extended inner automorphism $(\alpha_f)_f$ of G with $\alpha = \alpha_{\mathbf{id}_G}$.*

This provides a completely *element-free* characterization of inner automorphisms of groups! They are exactly those group automorphisms that are 'coherently extendible' along morphisms out of the domain.

Covariant Isotropy

- We have a functor $\mathcal{Z} : \textbf{Group} \to \textbf{Group}$ that sends any group G to its group of extended inner automorphisms $\mathcal{Z}(G)$. We refer to $\mathcal{Z}$ as the *covariant isotropy group (functor)* of the category **Group**.

- In fact, *any* category $\mathbb{C}$ has a *covariant isotropy group (functor)*

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \textbf{Group}$$

that sends each object $C \in \mathbb{C}$ to the group of extended inner automorphisms of C, i.e. families of automorphisms

$$\left(\alpha_f : \textbf{cod}(f) \xrightarrow{\sim} \textbf{cod}(f)\right)_{\textbf{dom}(f)=C}$$

in $\mathbb{C}$ with the same naturality property as before, i.e. natural automorphisms of the projection functor $C/\mathbb{C} \to \mathbb{C}$.

Covariant Isotropy

- We can also turn Bergman's characterization of inner automorphisms in **Group** into a *definition* of inner automorphisms in an arbitrary category $\mathbb{C}$: if $C \in \mathbb{C}$, we say that an automorphism $\alpha : C \xrightarrow{\sim} C$ is *inner* if there is an extended inner automorphism $(\alpha_f)_f \in \mathcal{Z}_{\mathbb{C}}(C)$ with $\alpha_{\mathbf{id}_C} = \alpha$.

- Notice that **Group** is the category of (set-based) *models* of an *algebraic theory*, i.e. a set of equational axioms between terms, namely the theory $\mathbb{T}_{\mathbf{Grp}}$ of groups. So $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$.

- We will generalize ideas from the proof of Bergman's Theorem to give a 'syntactic' characterization of the (extended) inner automorphisms of $\mathbb{T}\mathbf{mod}$, i.e. of the covariant isotropy group of $\mathbb{T}\mathbf{mod}$, for any so-called *quasi-equational* theory $\mathbb{T}$.

Quasi-Equational Theories

- What is a quasi-equational theory? (Also known as: partial Horn theory, essentially algebraic theory, cartesian theory, finite limit theory.)
- First, we need the notion of a *signature* Σ, which consists of a non-empty set $\Sigma_{\mathbf{Sort}}$ of *sorts*, and a set $\Sigma_{\mathbf{Fun}}$ of (typed) *function/operation symbols*.
- For example, the signature for *groups* has one sort X and three function symbols $\cdot : X \times X \to X$, $^{-1} : X \to X$, and $e : X$. The signature for *categories* has two sorts O, A and four function symbols $\mathbf{dom}, \mathbf{cod} : A \to O$, $\mathbf{id} : O \to A$, and $\circ : A \times A \to A$.

Quasi-Equational Theories

- We can then form the set $\textbf{Term}(\Sigma)$ of *terms* over Σ, constructed from variables and function symbols, as well as the set $\textbf{Horn}(\Sigma)$ of *Horn formulas* over Σ, which are finite conjunctions of equations between terms.

- A *quasi-equational theory* over a signature Σ is then a set of *implications* (the *axioms* of $\mathbb{T}$) of the form $\varphi \Rightarrow \psi$, with $\varphi, \psi \in \textbf{Horn}(\Sigma)$ (see [6]).

- The operation symbols of a quasi-equational theory are only required to be *partially* defined. If t is a term, we write $t \downarrow$ as an abbreviation for $t = t$, meaning 't is defined'.

Examples

- Any *algebraic* theory, whose axioms all have the form $\top \Rightarrow \psi$, where $\top$ is the empty conjunction. E.g. the theories of sets, semigroups, (commutative) monoids, (abelian) groups, (commutative) rings with unit, etc. For example, the theory $\mathbb{T}_{\mathbf{Grp}}$ of groups has the following axioms:

$$\top \Rightarrow x \cdot y \downarrow \wedge x^{-1} \downarrow \wedge e \downarrow,$$

$$\top \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$\top \Rightarrow x \cdot e = x \wedge \ e \cdot x = x,$$

$$\top \Rightarrow x \cdot x^{-1} = e \wedge \ x^{-1} \cdot x = e.$$

Examples

- The theories of categories, groupoids, categories with a terminal object, and cartesian (i.e. finitely complete) categories. E.g. two of the axioms of the theory of categories are

$$g \circ f \downarrow \Rightarrow \mathbf{dom}(g) = \mathbf{cod}(f),$$

$$\mathbf{dom}(g) = \mathbf{cod}(f) \Rightarrow g \circ f \downarrow .$$

- The theory of strict monoidal categories.

- The theory of functors $\mathcal{J} \to \mathbb{T}\mathbf{mod}$ for a small category $\mathcal{J}$ and quasi-equational theory $\mathbb{T}$. In particular, the theory of presheaves $\mathcal{J} \to \mathbf{Set}$.

Proof of Bergman's Theorem

- Let us focus on a specific idea in the proof of Bergman's Theorem.
- Consider the group $G\langle \mathbf{x} \rangle$ obtained from G by freely adjoining an indeterminate element $\mathbf{x}$. Elements of $G\langle \mathbf{x} \rangle$ are (reduced) group words in $\mathbf{x}$ and elements of G.
- The underlying set of $G\langle \mathbf{x} \rangle$ can be endowed with a *substitution monoid* structure: given $w_1, w_2 \in G\langle \mathbf{x} \rangle$, we set $w_1 \cdot w_2$ to be the reduction of $w_1[w_2/\mathbf{x}]$, and the unit is $\mathbf{x}$ itself.
- If $w \in G\langle \mathbf{x} \rangle$, w *commutes generically with* the group operations if:
  - In $G\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$, the reduction of $w[\mathbf{x}_1/\mathbf{x}]w[\mathbf{x}_2/\mathbf{x}]$ is $w[\mathbf{x}_1\mathbf{x}_2/\mathbf{x}]$;
  - In $G\langle \mathbf{x} \rangle$, the reduction of w^{-1} is $w[\mathbf{x}^{-1}/\mathbf{x}]$;
  - In $G\langle \mathbf{x} \rangle$, the reduction of $w[e/\mathbf{x}]$ in $G\langle \mathbf{x} \rangle$ is e.

Proof of Bergman's Theorem

- E.g. if $g \in G$, then the word $g\mathbf{x}g^{-1} \in G\langle \mathbf{x} \rangle$ commutes generically with the group operations:
  - $g\mathbf{x}_1 g^{-1} g\mathbf{x}_2 g^{-1} \sim g\mathbf{x}_1\mathbf{x}_2 g^{-1}$
  - $(g\mathbf{x}g^{-1})^{-1} \sim (g^{-1})^{-1}\mathbf{x}^{-1}g^{-1} \sim g\mathbf{x}^{-1}g^{-1}$,
  - $geg^{-1} \sim gg^{-1} \sim e$.
- Let $\mathcal{Z}(G)$ be the group of extended inner automorphisms of G, and let $\mathbf{Inv}(G\langle \mathbf{x} \rangle)$ be the subgroup of *invertible* elements of the substitution monoid $G\langle \mathbf{x} \rangle$. (E.g. $g\mathbf{x}g^{-1}$ is invertible, with inverse $g^{-1}\mathbf{x}g$.)
- Then the proof of Bergman's Theorem shows that the group $\mathcal{Z}(G)$ is isomorphic to the subgroup of $\mathbf{Inv}(G\langle \mathbf{x} \rangle)$ consisting of all words that commute generically with the group operations.

The Isotropy Group of a Quasi-Equational Theory

- Fix a quasi-equational theory $\mathbb{T}$ over a signature Σ, and let $\mathbb{T}\mathbf{mod}$ be the category of (set-based) models of $\mathbb{T}$. For simplicity, we will generally assume (in this talk) that $\mathbb{T}$ is single-sorted.

- We will now give a *logical/syntactic* characterization of the covariant isotropy group
$$\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\mathbf{mod} \to \mathbf{Group}$$
of $\mathbb{T}\mathbf{mod}$.

- Fix $M \in \mathbb{T}\mathbf{mod}$. As for groups, we can construct a $\mathbb{T}$-model $M\langle \mathbf{x} \rangle$, which is the coproduct of M with the free $\mathbb{T}$-model on one generator $\mathbf{x}$. Elements of $M\langle \mathbf{x} \rangle$ are (equivalence classes of) Σ-terms over $\mathbf{x}$ and elements of M. We can then endow the underlying set of $M\langle \mathbf{x} \rangle$ with a *substitution monoid* structure, in the same way as for groups.

The Isotropy Group of a Quasi-Equational Theory

In my thesis, I proved:

Theorem ([7])

Let $\mathbb{T}$ be a quasi-equational theory over a (single-sorted) signature Σ. For any $M \in \mathbb{T}\mathbf{mod}$, the covariant isotropy group $\mathcal{Z}_{\mathbb{T}}(M)$, i.e. the group of extended inner automorphisms of M, is isomorphic to the group of **invertible** *elements t of the substitution monoid $M\langle \mathbf{x} \rangle$ that* **commute generically with** *the function symbols of Σ, in the sense that if f is any n-ary function symbol of Σ, then*

$$t[f(\mathbf{x}_1, \ldots \mathbf{x}_n)/\mathbf{x}] = f(t[\mathbf{x}_1/\mathbf{x}], \ldots, t[\mathbf{x}_n/\mathbf{x}])$$

holds in $M\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$ (the coproduct of M with the free $\mathbb{T}$-model on n generators $\mathbf{x}_1, \ldots, \mathbf{x}_n$).

The Isotropy Group of a Quasi-Equational Theory

- In particular, an automorphism $\alpha : M \xrightarrow{\sim} M$ in $\mathbb{T}\mathbf{mod}$ is *inner* iff there is some $t \in \mathcal{Z}_{\mathbb{T}}(M)$ that *induces* α, i.e.

$$(m \in M) \qquad \alpha(m) = t[m/\mathbf{x}] \in M.$$

- Thus, Bergman's (syntactic) characterization of the (extended) inner automorphisms of $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$ extends to the category $\mathbb{T}\mathbf{mod}$ of (set-based) models of *any* quasi-equational theory $\mathbb{T}$.

Examples

- If $\mathbb{T}$ is the theory of sets, then $\mathbb{T}$ has trivial isotropy group, i.e. $\mathcal{Z}_{\mathbb{T}}(S) \cong \{\mathbf{x}\}$ for any set S, so the only inner automorphism of a set is the *identity* function.

- If $\mathbb{T}$ is the theory of groups, then Bergman proved $\forall G \in \mathbb{T}\mathbf{mod} = \mathbf{Group}$ that

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \{g\mathbf{x}g^{-1} \in G\langle\mathbf{x}\rangle \mid g \in G\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then $\forall M \in \mathbb{T}\mathbf{mod} = \mathbf{Mon}$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) \cong \{m\mathbf{x}m^{-1} \in M\langle\mathbf{x}\rangle \mid m \text{ is invertible in } M\} \cong \mathbf{Inv}(M).$$

Examples

- If $\mathbb{T}$ is the theory of abelian groups, then $\forall G \in \mathbb{T}\textbf{mod} = \textbf{Ab}$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \{\mathbf{x}, -\mathbf{x}\} \cong \mathbb{Z}_2.$$

- If $\mathbb{T}$ is the theory of commutative monoids or unital rings, then the isotropy group of $\mathbb{T}$ is trivial.

- If $\mathbb{T}$ is the theory of (not necessarily commutative) unital rings, then $\forall R \in \mathbb{T}\textbf{mod} = \textbf{Ring}$ we have

$$\mathcal{Z}_{\mathbb{T}}(R) \cong \{r\mathbf{x}r^{-1} \in R\langle \mathbf{x} \rangle \mid r \in R \text{ is a unit}\} \cong \textbf{Unit}(R).$$

- If $\mathbb{T}$ is the theory of categories, groupoids, or categories with a terminal object, then the isotropy group of $\mathbb{T}$ is trivial.

Examples

- If $\mathbb{T}$ is the theory of strict monoidal categories, then for any strict monoidal category $\mathbb{C}$ we have

$$\mathcal{Z}_{\mathbb{T}}(\mathbb{C}) \cong \mathbf{Inv}\left(\mathbb{C}_O, \otimes^{\mathbb{C}}, e^{\mathbb{C}}\right),$$

the group of invertible elements of the object monoid $\left(\mathbb{C}_O, \otimes^{\mathbb{C}}, e^{\mathbb{C}}\right)$ of $\mathbb{C}$. In particular, if $F : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ is a (strict monoidal) automorphism of a strict monoidal category $\mathbb{C}$, then F is *inner* iff there is some invertible object $c \in \mathbb{C}$ such that F is given by *conjugation* with c, i.e.

$$(a \in \mathbb{C}_O) \qquad F(a) = c \otimes a \otimes c^{-1}$$

and

$$(f \in \mathbb{C}_A) \qquad F(f) = \mathbf{id}_c \otimes f \otimes \mathbf{id}_{c^{-1}}.$$

Isotropy Groups of Functor Categories

- We can also characterize the covariant isotropy groups of *functor categories* of the form $\mathbb{T}\textbf{mod}^{\mathcal{J}}$, for a quasi-equational theory $\mathbb{T}$ and small category $\mathcal{J}$. In particular, we can characterize the covariant isotropy groups of presheaf categories $\textbf{Set}^{\mathcal{J}}$.

- Fix a quasi-equational theory $\mathbb{T}$. Given a small category $\mathcal{J}$, we can define a quasi-equational theory $\mathbb{T}^{\mathcal{J}}$ whose models are functors $\mathcal{J} \to \mathbb{T}\textbf{mod}$, i.e.

$$\mathbb{T}^{\mathcal{J}}\textbf{mod} \cong \mathbb{T}\textbf{mod}^{\mathcal{J}}.$$

Isotropy Groups of Functor Categories

In my thesis, I then proved the following theorem:

Theorem ([7])

Let $\mathbb{T}$ be a (single-sorted) quasi-equational theory (satisfying a few technical assumptions), and let $\mathcal{J}$ be a small category, with $\mathbf{Aut}(\mathbf{Id}_{\mathcal{J}})$ *the group of natural automorphisms of* $\mathbf{Id}_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ *(which we may call the* **global isotropy group** *of* $\mathcal{J}$*). For any functor* $F : \mathcal{J} \to \mathbb{T}\mathbf{mod}$*, we have*

$$\mathcal{Z}_{\mathbb{T}\mathbf{mod}^{\mathcal{J}}}(F) \cong \mathbf{lim}(\mathcal{Z}_{\mathbb{T}} \circ F) \times \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}}) \in \mathbf{Group}.$$

In particular, for any functor $F : \mathcal{J} \to \mathbf{Set}$*, we have*

$$\mathcal{Z}_{\mathbf{Set}^{\mathcal{J}}}(F) \cong \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}}).$$

Isotropy Groups of Functor Categories

- In particular, if $F : \mathcal{J} \to \mathbf{Set}$ is a functor and $\alpha : F \xrightarrow{\sim} F$ is an automorphism, then α is *inner* iff there is some $\psi \in \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}})$ with

$$(k \in \mathcal{J}) \qquad \alpha_k = F(\psi_k) : F(k) \xrightarrow{\sim} F(k).$$

- So the covariant isotropy group functor $\mathcal{Z} : \mathbf{Set}^{\mathcal{J}} \to \mathbf{Group}$ is *constant* on the global isotropy group $\mathbf{Aut}(\mathsf{Id}_{\mathcal{J}})$ of $\mathcal{J}$.

Isotropy Groups of G-Sets

- For any group G, the covariant isotropy group functor $\mathcal{Z} : \mathbf{Set}^G \to \mathbf{Group}$ of the category of G-sets is *constant* on the centre $Z(G)$ of the group G.

- More generally, for any monoid M, the covariant isotropy group functor $\mathcal{Z} : \mathbf{Set}^M \to \mathbf{Group}$ of the category of M-sets is *constant* on the group $\mathbf{Inv}(Z(M))$ of invertible elements of the centre of M.

Connections with Topos Theory

- If $\mathbb{T}$ is a quasi-equational theory, then $\mathbb{T}$ has a *classifying topos* $\mathcal{B}(\mathbb{T})$, which is a cocomplete topos that has a *universal model* of $\mathbb{T}$ and classifies all topos-theoretic models of $\mathbb{T}$ ([4], [5]).
- It has been shown that any Grothendieck topos $\mathcal{E}$ has a canonical internal group object called the *isotropy group* of the topos, which acts canonically on every object of the topos and formally generalizes the notion of conjugation ([3]).
- The covariant isotropy group $\mathcal{Z}_{\mathbb{T}}$ of a quasi-equational theory $\mathbb{T}$ *is* in fact the isotropy group object of the classifying topos $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$ ([3], [4]).

Conclusions

- Bergman's *element-free* characterization of the inner automorphisms of groups can be used to *define* inner automorphisms in arbitrary categories.
- We have extended Bergman's *syntactic* characterization of the (extended) inner automorphisms of groups, i.e. of the covariant isotropy group of $\mathbf{Group} = \mathbb{T}_{\mathsf{Grp}}\mathbf{mod}$, to the covariant isotropy group of $\mathbb{T}\mathbf{mod}$ for *any* quasi-equational theory $\mathbb{T}$.
- Using this characterization, we have obtained concrete descriptions of the (extended) inner automorphisms in several different categories: $\mathbf{Set}, \mathbf{Group}, \mathbf{Mon}, \mathbf{Ab}, \mathbf{Ring}, \mathbf{Cat}, \mathbf{StrMonCat}, \mathbb{T}\mathbf{mod}^{\mathcal{J}}, \mathbf{Set}^{\mathcal{J}}, \ldots$
- This work also represents a contribution to the more general project of characterizing the isotropy group objects of Grothendieck toposes.

Some Future Directions

- Given (disjoint) theories $\mathbb{T}_1$ and $\mathbb{T}_2$, characterize the covariant isotropy group of the category of models of $\mathbb{T}_1$ in $\mathbb{T}_2$mod (i.e. the category of models of $\mathbb{T}_1 \otimes \mathbb{T}_2$) in terms of the covariant isotropy groups of $\mathbb{T}_1$ and $\mathbb{T}_2$ (subsuming the examples of strict monoidal categories and functor categories $\mathbb{T}\mathbf{mod}^{\mathcal{J}}$).

- Characterize the covariant isotropy groups of Grothendieck toposes, i.e. categories $\mathsf{Sh}(\mathbb{C}, J)$ in terms of the (small) site presentation $(\mathbb{C}, J)$. Categories of the form $\mathsf{Sh}(\mathbb{C}, J)$ are categories of models for an (infinitary) quasi-equational theory.

- Characterize the *contravariant* isotropy groups of quasi-equational theories.

Thank you!

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