Varieties of Heyting algebras: what we (still don't) know

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Definition

- A Heyting algebra is a bounded lattice A such that $\wedge : A^2 \to A$ has a residual $\to : A^2 \to A$ given by $a \wedge x \leq b \iff x \leq a \to b$.
- The class $\mathbb{HA}$ of Heyting algebras is equationally definable, hence forms a variety.
- Heyting algebras are rather non-symmetric objects that pop up in different branches of mathematics.

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- Heyting algebras are rather non-symmetric objects that pop up in different branches of mathematics.

Examples from logic

- A standard example of a Heyting algebra is the Lindenbaum algebra $\mathcal{L}$ of the intuitionistic calculus.
- In fact, $\mathcal{L}$ generates $\mathbb{HA}$.
- An intermediate logic is a logic between intuitionistic and classical logics (Umezawa, 1950s).
- The Lindenbaum algebra of each intermediate logic is a Heyting algebra.
- The lattice of intermediate logics is dually isomorphic to the lattice of nontrivial varieties of Heyting algebras.

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Other examples

- Topology: The opens $\mathcal{O}(X)$ of each topological space X form a Heyting algebra, where $U \to V = \mathrm{int}(U^c \cup V)$.
- Kripke semantics: In particular, the upsets $\mathrm{Up}(X)$ of each poset X form a Heyting algebra.
- Locale theory: Every locale is a Heyting algebra. In fact, a complete lattice is a Heyting algebra iff it is a locale (satisfies the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$). Domain theory: Every continuous distributive lattice is a Heyting algebra.
- Topos theory: The subobject classifier in every topos is a Heyting algebra.
- Universal algebra: Every algebraic distributive lattice is a Heyting algebra. Thus, the congruence lattice of every algebra in a congruence-distributive variety is a Heyting algebra.

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Interior operators and Heyting algebras

- An interior operator on a boolean algebra is a unary function $\Box : B \to B$ that satisfies the Kuratowski axioms: $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box a \leq a$, and $\Box a \leq \Box\Box a$.
- The fixpoints $\mathrm{Fix}(\Box)$ form a bounded sublattice of B which is a Heyting algebra, where $a \to b = \Box(a^* \vee b)$. Moreover, a bounded sublattice L of B is a Heyting algebra iff the embedding $L \hookrightarrow B$ has a right adjoint.
- Let $\mathbb{IA}$ be the variety of interior algebras. Associating with each $(B, \Box) \in \mathbb{IA}$ the fixpoints $\mathrm{Fix}(\Box)$ defines a functor $F : \mathbb{IA} \to \mathbb{HA}$.

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Boolean envelopes

- This functor has a left adjoint $L : \mathbb{HA} \to \mathbb{IA}$.
- For a Heyting algebra A, let B_A be the boolean envelope of A (the boolean algebra freely generated by A).
- We can write each element of B_A in the conjunctive normal form: $x = \bigwedge_{i=1}^{n}(a_i^* \vee b_i)$, where $a_i, b_i \in A$ and a_i^* is the complement of a_i in B_A.
- Define $\Box_A x = \bigwedge_{i=1}^{n}(a_i \to b_i)$, where $\to$ is the implication in A.
- Then $(B_A, \Box_A) \in \mathbb{IA}$ and this correspondence extends to a functor $L : \mathbb{HA} \to \mathbb{IA}$ that is left adjoint to $F : \mathbb{IA} \to \mathbb{HA}$.

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Modal companions

- For a variety $\mathbb{U}$ of interior algebras, the class $F(\mathbb{U}) = \{F(B, \Box) \mid (B, \Box) \in \mathbb{U}\}$ is a variety of Heyting algebras.
- Let $\mathbb{V}$ be a variety of Heyting algebras. We call a variety $\mathbb{U}$ of interior algebras a modal companion of $\mathbb{V}$ if $F(\mathbb{U}) = \mathbb{V}$.
- Each $\mathbb{V}$ has many modal companions (often continuum many).
- $F^{-1}(\mathbb{V}) := \{(B, \Box) \mid F(B, \Box) \in \mathbb{V}\}$ is the largest modal companion of $\mathbb{V}$ (Gödel translation).
- The class $L(\mathbb{V}) = \{L(A) \mid A \in \mathbb{V}\}$ may not be a variety of interior algebras (may not be closed under products).
- Let $L^*(\mathbb{V})$ be the variety generated by $L(\mathbb{V})$.
- Then $L^*(\mathbb{V})$ is the least modal companion of $\mathbb{V}$.

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Hausdorff residue and Grzegorczyk algebras

- Let $\Diamond$ be the closure operator defined by $\Diamond a = (\Box a^*)^*$.
- The Hausdorff residue is defined by $\rho(a) = a \wedge \Diamond(\Diamond a \wedge a^*)$.
- Define $\rho^{k+1}(a) = \rho(\rho^k(a))$ and call a cyclic if $a \neq 0$ and $\rho^k(a) = \rho^{k+1}(a) \neq 0$ for some k.
- An interior algebra is a Grzegorczyk algebra if it has no cyclic elements.

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Blok-Esakia theorem

- The variety $\mathbb{GRZ}$ of Grzegorczyk algebras is exactly the variety generated by $L(\mathbb{HA})$.
- Therefore, $\mathbb{GRZ}$ is the least modal companion of $\mathbb{HA}$.
- The modal companions of $\mathbb{V}$ form the interval $[L^*(\mathbb{V}), F^{-1}(\mathbb{V})]$ in the lattice of subvarieties of $\mathbb{IA}$.
- L^* is an isomorphism from the lattice of varieties of Heyting algebras to the lattice of varieties of Grzegorczyk algebras (Blok, Esakia, 1976).

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Priestley duals of Heyting algebras

- For a Heyting algebra A, let X_A be the poset of prime filters of A (ordered by inclusion).
- For $a \in A$, let $\varphi(a) = \{x \in X_A \mid a \in x\}$ (Stone map).
- The Priestley topology on X_A is given by the subbasis $\{\varphi(a) \mid a \in A\} \cup \{\varphi(a)^c \mid a \in A\}$.
- This is a Stone topology on X_A (zero-dimensional, compact, Hausdorff). Moreover, if $x \not\leq y$, then x can be separated from y by a clopen upset (Priestley separation).
- Since A is a Heyting algebra, the order is continuous (the downset of clopen is clopen).

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Esakia spaces

- An Esakia space is a Stone space X equipped with a continuous partial order $\leq$.
- An Esakia morphism is a continuous map $f : X \to Y$ such that $\uparrow f(x) = f[\uparrow x]$ (p-morphism)
- Let $\mathbb{ES}$ be the category of Esakia spaces and Esakia morphisms.
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Jónsson-Tarski duality for interior algebras

- For an interior algebra $(B, \Box)$, let X_B be the Stone space (space of ultrafilters) of B.
- Define $\sqsubseteq$ on X_B by $x \sqsubseteq y$ iff $\Box^{-1}[x] \subseteq y$ ($\Box a \in x$ implies $a \in y$).
- Then $\sqsubseteq$ is a continuous pre-order on X_B.
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- Call an interior algebra $(B, \Box)$ skeletal if B is the Boolean envelope of the Heyting algebra $\mathrm{Fix}(\Box)$ of fixpoints of $\Box$.
- $(B, \Box)$ is skeletal iff X_B is an Esakia space (so the preorder $\sqsubseteq$ on X_B is a partial order).
- The category $\mathbb{SA}$ of skeletal algebras is equivalent to $\mathbb{HA}$.
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Hosoi slices

- The depth of a Heyting algebra A is the maximal length of chains in X_A provided the max is finite. Otherwise it is infinite.
- The depth of a variety $\mathbb{V}$ of Heyting algebras is $\leq n$ if n bounds the depths of all $A \in \mathbb{V}$. Otherwise it is infinite.
- For each n, there exist the least and greatest varieties of depth n. The least one is the variety $\mathbb{C}_n$ generated by the $(n+1)$-chain and the greatest is the variety $\mathbb{HA}_n$ of all Heyting algebras of depth n.
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Cardinality of $\mathcal{S}_n$

- $\mathcal{S}_0$ only consists of the trivial variety.
- $\mathcal{S}_1$ only consists of the variety $\mathbb{BA} = \mathbb{C}_1 = \mathbb{HA}_1$.
- $\mathcal{S}_2$ is countable.
- Let A_n be the Heyting algebra obtained by adjoining a new top to the boolean algebra 2^n. Its dual space is the n-fork.
- Let $\mathbb{V}_n = \mathsf{HSP}(A_n)$. Then $\mathcal{S}_2$ is the chain

$$\mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset \mathbb{V}_n \subset \cdots \subset \mathsf{HSP}\{A_n \mid n \geq 1\}$$

- If $n \geq 3$, then $\mathcal{S}_n$ is uncountable.

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Locally finite varieties

- A variety $\mathbb{V}$ is of finite depth if $\mathbb{V} \in \mathcal{S}_n$ for some n.
- Every variety of finite depth is locally finite (Kuznetsov, Komori, 1970s). However, there exist locally finite varieties of infinite depth.
- For example, the variety $\mathbb{LC}$ of all chains is locally finite.
- A characterization of locally finite varieties of Heyting algebras remains elusive.
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Mardaev's theorem

- Roughly speaking, a Brouwerian algebra is a Heyting algebra without bottom in the signature.
- Mardaev's theorem: Each n-generated Brouwerian algebra embeds into a 2-generated Brouwerian algebra.
- But the embedding does not preserve 0. And there's now evidence suggesting that it is likely that for each n there exists a non-locally finite variety $\mathbb{V}$ of Heyting algebras such that the free n-generated $\mathbb{V}$-algebra is finite.
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Kuznetsov's hierarchy

- $\mathcal{FG}$ = finitely generated varieties (generated by one finite algebra).
- $\mathcal{LF}$ = locally finite varieties.
- $\mathcal{FMP}$ = varieties generated by finite algebras.
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- Kuznetsov's problem: It also remains open if there exist varieties (logics) that are topologically incomplete.
- It is also open whether there exist varieties that are not generated by complete Heyting algebras.
- Whether being generated by complete Heyting algebras implies topological completeness is also open.

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Related results

- All these classes become distinguishable in the signature of modal algebras.
- In particular, Kuznetsov's problem has a negative solution for varieties of interior algebras (Shehtman).
- A Heyting algebra A is a bi-Heyting algebra if its order-dual A^d is also a Heyting algebra.
- Recent result: Kuznetsov's problem has a negative solution for varieties of bi-Heyting algebras (jointly with Gabelaia and Jibladze).
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$\mathcal{P}$-DEGREES

- Let $\mathcal{P}$ be a property. For a cardinal κ and variety $\mathbb{V}$, we say that the $\mathcal{P}$-degree of $\mathbb{V}$ is κ provided there are κ-many varieties that share the property $\mathcal{P}$ with $\mathbb{V}$.
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Degrees of fmp

- We know that there are continuum many Kripke incomplete varieties of Heyting algebras. But we know little about the degree of Kripke incompleteness for varieties of Heyting algebras.
- The situation becomes drastically different from Blok's dichotomy theorem if we ask about the degree of fmp.
- Antidichotomy theorem: If $\kappa \geq 1$ is countable or $2^{\aleph_0}$, then κ is realized as the degree of fmp of some variety $\mathbb{V}$ of Heyting algebras. Assuming CH, every cardinal $1 \leq \kappa \leq 2^{\aleph_0}$ is realized as the degree of fmp of some $\mathbb{V}$ (jointly with my brother Nick and Tommaso Moraschini).

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Thank you!