# Equational theories of idempotent semifields

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Joint work with Simon Santschi

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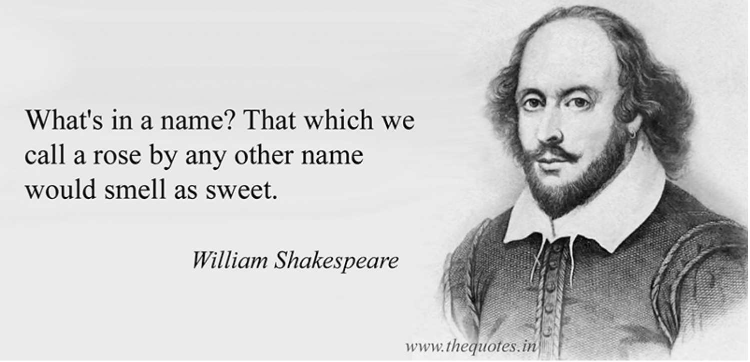
# A Rose by any other Name?

An idempotent semifield is a lattice-ordered group ($\ell$-group) $\langle L, \not\wedge, \vee, \cdot, \not{}^{-1}, \mathrm{e}\rangle$ without the inverse and meet operations.



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We will see that . . .

- *unlike* $\ell$-groups, no non-trivial class of idempotent semifields is finitely based;
- *like* $\ell$-groups, there are continuum-many equational theories of classes of idempotent semifields;
- *like* $\ell$-groups, the equational theory of the class of idempotent semifields is co-NP-complete.

Equational theories of idempotent semifields.
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# Idempotent Semifields

An idempotent semiring is an algebraic structure $\langle S, \vee, \cdot, \mathrm{e} \rangle$ satisfying

(i) $\langle S, \cdot, \mathrm{e} \rangle$ is a monoid;

(ii) $\langle S, \vee \rangle$ is a semilattice, where $a \leq b :\Longleftrightarrow a \vee b = b$;

(iii) $a(b \vee c)d = abd \vee acd$ for all $a, b, c, d \in S$.

If $\langle S, \cdot, \mathrm{e} \rangle$ is a group, this structure is an idempotent semifield, and adding the operations $^{-1}$ (inverse) and $\wedge$ (meet) yields an $\ell$-group.

**Remark**

Alternative definitions of idempotent semirings (*dioids* / *ai-semirings*) differ with respect to the presence of constants $\mathrm{e}$ and 0 in the signature.

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# Commutative Examples

- $\langle \mathbb{Z}, \mathsf{max}, +, 0 \rangle$ is a commutative idempotent semifield, but its subalgebra $\langle \mathbb{N}, \mathsf{max}, +, 0 \rangle$ is only an idempotent semiring, so idempotent semifields do not form a variety.
- Every ordered commutative idempotent semifield embeds into a lexicographically ordered power of $\langle \mathbb{R}, \mathsf{max}, +, 0 \rangle$ (Hahn 1907).
- More generally, every commutative idempotent semifield embeds into an idempotent semifield of real-valued functions on a poset (Conrad et al. 1963).

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# Fundamental Examples

The order-preserving bijections on any chain $\mathbf{\Omega} = \langle \Omega, \preceq \rangle$ equipped with function composition form an idempotent semifield $\mathbf{Aut}(\mathbf{\Omega})$, where

$$f \leq g \ :\Longleftrightarrow\ f(a) \preceq g(a) \text{ for all } a \in \Omega.$$

Moreover, every idempotent semifield embeds into $\mathbf{Aut}(\mathbf{\Omega})$ for some chain $\Omega$ (Holland 1963), and the equational theory of the class of all idempotent semifields is the equational theory of $\mathbf{Aut}(\langle \mathbb{R}, \leq \rangle)$.

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# Our First Question

Which classes of idempotent semifields are finitely based ?

# The Finite Basis Problem

Let K be any class of algebras over some fixed signature, and call it non-trivial if at least one of its members has more than one element.

A basis for its equational theory $\mathrm{Eq}(\mathsf{K})$ is a subset $\Sigma \subseteq \mathrm{Eq}(\mathsf{K})$ such that $\Sigma \models \mathrm{Eq}(\mathsf{K})$; if $\mathrm{Eq}(\mathsf{K})$ has a finite basis, K is said to be finitely based.

E.g., the $\ell$-group $\langle \mathbb{Z}, \min, \max, +, -, 0 \rangle$ is finitely based, but not the idempotent semifield $\langle \mathbb{Z}, \max, +, 0 \rangle$ (Aceto et al. 2003); indeed, there are infinitely many finitely based varieties of $\ell$-groups, but . . .

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We define the flat extension of a monoid $\mathbf{M} = \langle M, \cdot, \mathrm{e} \rangle$ as

$$\flat(\mathbf{M}) = \langle M \cup \{\top\}, \vee, \star, \mathrm{e} \rangle,$$

where $\top \notin M$ and for all $a, b \in M \cup \{\top\}$,

$$a \star b := \begin{cases} a \cdot b & \text{if } a, b \in M \\ \top & \text{otherwise;} \end{cases} \qquad a \vee b := \begin{cases} a & \text{if } a = b \\ \top & \text{otherwise.} \end{cases}$$

Then $\langle M \cup \{\top\}, \star, \mathrm{e} \rangle$ is a monoid, $\langle M \cup \{\top\}, \vee \rangle$ is a semilattice, and

$$\flat(\mathbf{M}) \text{ is an idempotent semiring} \iff \mathbf{M} \text{ is cancellative.}$$

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# The Key Examples

The cyclic groups

$$\mathbf{Z} = \langle \mathbb{Z}, +, 0 \rangle, \qquad \mathbf{Z}_n = \langle \{\mathrm{e}, a, \ldots, a^{n-1}\}, \cdot, \mathrm{e} \rangle \ \ (n \in \mathbb{N}^+)$$

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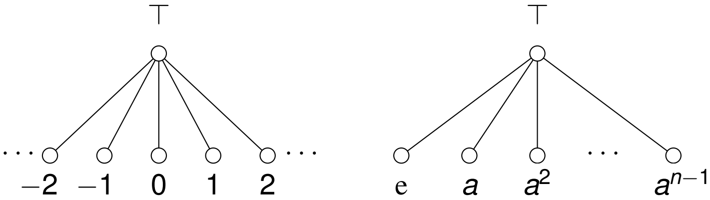

# Semiring Inequations and Monoid Quasiequations

Given $\varepsilon = (s \leq t_1 \vee \cdots \vee t_n)$, where $s, t_1, \ldots, t_n$ are monoid terms such that each variable occurring in $s$ occurs in $t_1 \vee \cdots \vee t_n$, define

$$Q(\varepsilon) := \{t_1 \approx t_2, \ldots, t_1 \approx t_n\} \Rightarrow t_1 \approx s.$$

**Lemma**

*For any monoid* $\mathbf{M}$ *and equation* $\varepsilon$ *of the above 'suitable' form:*

$$\flat(\mathbf{M}) \models \varepsilon \iff \mathbf{M} \models Q(\varepsilon).$$

For example, consider for any $n > 1$,

$$\varepsilon = (x \leq \mathrm{e} \vee x^n) \quad \text{and} \quad Q(\varepsilon) = (\mathrm{e} \approx x^n \Rightarrow \mathrm{e} \approx x),$$

and observe that $\mathbf{Z} \models Q(\varepsilon)$, so $\flat(\mathbf{Z}) \models \varepsilon$, but $\mathbf{Z}_n \not\models Q(\varepsilon)$, so $\flat(\mathbf{Z}_n) \not\models \varepsilon$.

# Semiring Inequations and Monoid Quasiequations

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$$Q(\varepsilon) := \{t_1 \approx t_2, \ldots, t_1 \approx t_n\} \Rightarrow t_1 \approx s.$$

**Lemma**

*For any monoid* $\mathbf{M}$ *and equation* $\varepsilon$ *of the above 'suitable' form:*

$$\flat(\mathbf{M}) \models \varepsilon \iff \mathbf{M} \models Q(\varepsilon).$$

For example, consider for any $n > 1$,

$$\varepsilon = (x \le \mathrm{e} \vee x^n) \quad \text{and} \quad Q(\varepsilon) = (\mathrm{e} \approx x^n \Rightarrow \mathrm{e} \approx x),$$

and observe that $\mathbf{Z} \models Q(\varepsilon)$, so $\flat(\mathbf{Z}) \models \varepsilon$, but $\mathbf{Z}_n \not\models Q(\varepsilon)$, so $\flat(\mathbf{Z}_n) \not\models \varepsilon$.

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# The Crucial Lemma

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*For any conjunction of monoid quasiequations* $\alpha$*,*

$$\mathbf{Z} \models \alpha \iff \exists n \in \mathbb{N} \colon \mathbf{Z}_p \models \alpha \text{ for each prime } p > n.$$

## Proof sketch.

($\Leftarrow$) $\mathbf{Z}$ embeds into $\prod\{\mathbf{Z}_p \mid p > n \text{ is prime}\}$.

($\Rightarrow$) Suppose that $\mathbf{Z} \models \alpha$. Then $T \cup \{\{x^k \approx \mathrm{e}\} \Rightarrow x \approx \mathrm{e} \mid k \in \mathbb{N}\} \models \alpha$, where $T$ is a first-order axiomatization of the class of Abelian groups. By compactness, $T \cup \{x^2 \approx \mathrm{e} \Rightarrow x \approx \mathrm{e}, \ldots, x^n \approx \mathrm{e} \Rightarrow x \approx \mathrm{e}\} \models \alpha$ for some $n \in \mathbb{N}$. Hence $\mathbf{Z}_p \models \alpha$ for each prime $p > n$. □

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# The Finite Basis Theorem

**Theorem (M. & Santschi 2025)**

*No non-trivial class of idempotent semifields is finitely based.*

**Proof sketch.**

Let K be any finitely based class of idempotent semifields, and w.l.o.g. let $\Sigma$ be a finite basis of 'suitable inequations' for $\mathrm{Eq}(\mathsf{K})$.

For any prime $p$: $\mathsf{K} \models x \leq \mathrm{e} \vee x^p$, but $\flat(\mathbf{Z}_p) \not\models x \leq \mathrm{e} \vee x^p$, so $\flat(\mathbf{Z}_p) \not\models \Sigma$.

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# A Second Question

How many equational theories of idempotent semifields are there ?

# The Answer

**Theorem (M. & Santschi 2025)**

*There are continuum-many equational theories of classes of idempotent semifields.*

Let us call a variety of $\ell$-groups defined by the equational theory of ordered groups together with a set of group equations g-representable.

**Theorem (Kopytov & Medvedev 1977)**

*There are continuum-many g-representable varieties of $\ell$-groups.*

To prove our theorem, it suffices to show that any two g-representable varieties of $\ell$-groups can be distinguished by a semiring equation.

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# Splitting Products

Every g-representable variety V of $\ell$-groups has the following product-splitting property: for any terms $s, t, u$ and new variable $x$:

$$\mathsf{V} \models \mathrm{e} \leq u \vee st \iff \mathsf{V} \models \mathrm{e} \leq u \vee s\bar{x} \vee xt. \quad (\bar{x} := x^{-1})$$

For such a variety, we can 'eliminate inverses' from inequations, e.g.,

$$\begin{aligned}
\mathsf{V} \models \mathrm{e} \leq v\bar{z}v \vee z\bar{v} &\iff \mathsf{V} \models \mathrm{e} \leq v\bar{z}\bar{x} \vee xv \vee z\bar{v} \\
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&\iff \mathsf{V} \models xz \leq v \vee xvxz \vee z\bar{v}\bar{y} \vee xzyxz \\
&\iff \mathsf{V} \models xzyv \leq vyv \vee xvxzyv \vee z \vee xzyxzyv.
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# The Counting Theorem

**Theorem (M. & Santschi 2025)**

*There are continuum-many equational theories of classes of idempotent semifields.*

**Proof sketch.**

Let $\mathsf{V}_1$ and $\mathsf{V}_2$ be distinct g-representable varieties of $\ell$-groups. W.l.o.g., $\mathsf{V}_1 \models \varepsilon$ and $\mathsf{V}_2 \not\models \varepsilon$ for some inequation $\varepsilon = (\mathrm{e} \leq t_1 \vee \cdots \vee t_n)$, where $t_1, \ldots, t_n$ are group terms. However, since $\mathsf{V}_1$ and $\mathsf{V}_2$ have the product-splitting property, eliminating inverses produces a semiring inequation $\varepsilon^\star$ such that $\mathsf{V}_1 \models \varepsilon^\star$ and $\mathsf{V}_2 \not\models \varepsilon^\star$. That is, the idempotent semifield reducts of $\mathsf{V}_1$ and $\mathsf{V}_2$ have distinct equational theories. □

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# A Third Question

How hard is deciding equations in the class of idempotent semifields?

# Towards an Answer

- The equational theory of $\ell$-groups is known to be co-NP-complete (Galatos & M. 2016), so the equational theory of idempotent semifields belongs to co-NP; but is it co-NP-hard?
- The equational theory of Abelian $\ell$-groups is co-NP-complete, but deciding inequations $s \leq t_1 \vee \cdots \vee t_n$, where $s, t_1, \ldots, t_n$ are monoid (or group) terms, in this variety belongs to P.
- We will see that deciding inequations $s \leq t_1 \vee \cdots \vee t_n$, where $s, t_1, \ldots, t_n$ are monoid terms, in the class of idempotent semifields (equivalently, the variety of $\ell$-groups) is co-NP-hard.

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# Splitting Meets

The variety LG of $\ell$-groups has the product-splitting property and also the meet-splitting property: for any terms $s, t, u$ and new variable $x$,

$$\mathsf{LG} \models \mathrm{e} \leq u \vee (s \wedge t) \iff \mathsf{LG} \models \mathrm{e} \leq u \vee sx \vee t\bar{x}.$$

For example,

$$\mathsf{LG} \models \mathrm{e} \leq y \vee (\bar{y} \wedge z\bar{y}\bar{z}) \iff \mathsf{LG} \models \mathrm{e} \leq y \vee \bar{y}x \vee z\bar{y}\bar{z}\bar{x}$$
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**Remark**

No non-trivial proper variety of $\ell$-groups has this property; such a variety would satisfy $\mathrm{e} \leq (y \vee \mathrm{e})^2\bar{x} \vee (y \vee \mathrm{e})^{-1}x \vee (y \vee \mathrm{e})^{-1}$ and hence, by the meet-splitting property, $\mathrm{e} \leq (y \vee \mathrm{e})^{-1}$, yielding triviality.

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$$\begin{aligned} \mathsf{LG} \models \mathrm{e} \leq y \vee (\bar{y} \wedge z\bar{y}\bar{z}) &\iff \mathsf{LG} \models \mathrm{e} \leq y \vee \bar{y}x \vee z\bar{y}\bar{z}\bar{x} \\ &\iff \mathsf{LG} \models yxzy \leq y^2xzy \vee x^2zy \vee yz. \end{aligned}$$

## Remark

No non-trivial proper variety of $\ell$-groups has this property; such a variety would satisfy $\mathrm{e} \leq (y \vee \mathrm{e})^2\bar{x} \vee (y \vee \mathrm{e})^{-1}x \vee (y \vee \mathrm{e})^{-1}$ and hence, by the meet-splitting property, $\mathrm{e} \leq (y \vee \mathrm{e})^{-1}$, yielding triviality.

# A Hardness Lemma

## Lemma

*Deciding inequations $s \leq t_1 \vee \cdots \vee t_n$, where $s, t_1, \ldots, t_n$ are monoid terms, in the variety* LG *of $\ell$-groups is co-NP-hard.*

## Proof sketch.

Deciding equations of the following form in the variety DL of distributive lattices is co-NP-hard (Hunt, Rosenkrantz, Bloniarz 1987):

$$\varepsilon = \bigwedge_{i \in I} \bigvee_{j \in J_i} x_{ij} \leq \bigvee_{k \in K} \bigwedge_{l \in L_k} y_{kl} \vee z_{kl}, \quad \text{where the } x_{ij}, y_{kl}, z_{kl} \text{ are variables.}$$

Hence it suffices to use the meet-splitting property to produce for input $\varepsilon$ (in polynomial time) a suitable inequation $\delta$ of size polynomial in the size of $\varepsilon$ such that $\mathsf{DL} \models \varepsilon \iff \mathsf{LG} \models \delta$. □

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# The Complexity Theorem

**Theorem (M. & Santschi 2025)**

*The equational theory of the class of idempotent semifields is co-NP-complete.*

**Remark**

The equational theory of commutative idempotent semifields is also co-NP-complete, but the decidability of the equational theory of ordered idempotent semifields (or, similarly, ordered groups) is open.

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# Coda: Left-Orders

A left-order on a monoid **M** is a total order $\leq$ on $M$ satisfying

$$x \leq y \implies zx \leq zy.$$

Let $\mathbf{FG}(X)$ denote the free group over a set $X$.

**Theorem (Colacito & M. 2019)**

*The following are equivalent for any $t_1, \ldots, t_n \in \mathrm{FG}(X)$:*

(1) *There exists a left-order $\leq$ on $\mathbf{FG}(X)$ such that $\mathrm{e} < t_1, \ldots, \mathrm{e} < t_n$.*

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# Summing Up

- Idempotent semifields are reducts of $\ell$-groups, but their equational theories have different properties and require new proof methods.
- There are continuum-many equational theories of classes of idempotent semifields, but only the trivial theory has a finite basis.
- The equational theory of idempotent semifields is co-NP-complete and deciding if there exists a left-order on a free group or monoid satisfying a given finite set of inequalities is NP-complete.

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# Open Problems

1. Can we describe a broader family of classes of idempotent semirings that are not finitely based?

2. Is the equational theory of totally ordered idempotent semifields — or, similarly, ordered groups — decidable?

   This problem is equivalent to deciding if there exists a bi-order on a free group satisfying a given finite set of inequalities.

3. Is the equational theory of totally ordered commutative idempotent semirings decidable?

   This problem amounts to deciding if there exists a total preorder on $\langle \mathbb{N}^k, +, \bar{0} \rangle$ satisfying $\bar{u} < \bar{v}_1, \ldots, \bar{u} < \bar{v}_n$ with $\bar{u}, \bar{v}_1, \ldots, \bar{v}_n \in \mathbb{N}^k$.

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