

Almost Symmetric Linear Arc Monadic Datalog and Transitive Tournaments

Florian Starke

joint work with Sebastian Meyer

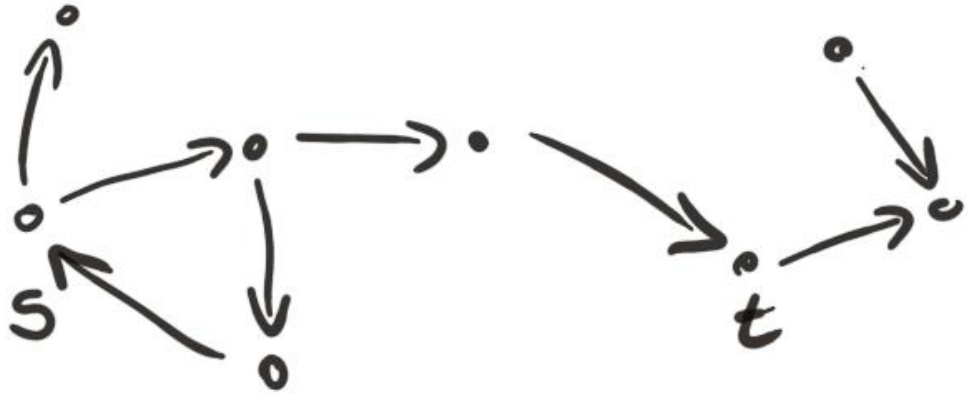




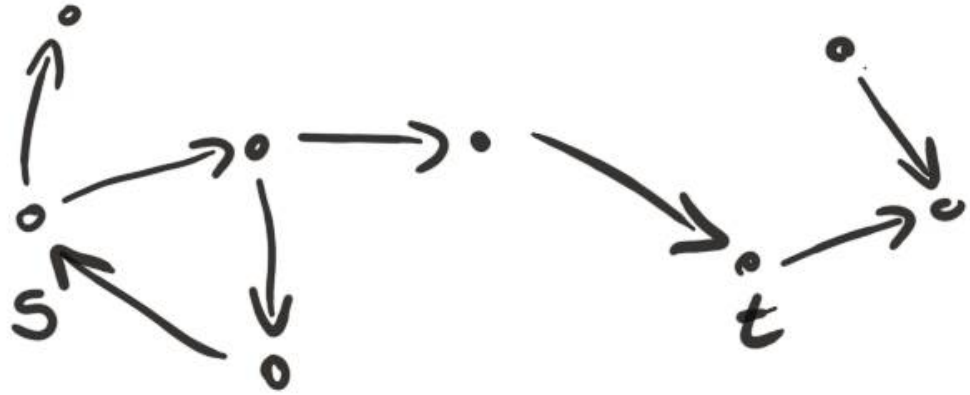
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Directed Reachability and Datalog

Directed Reachability and Datalog



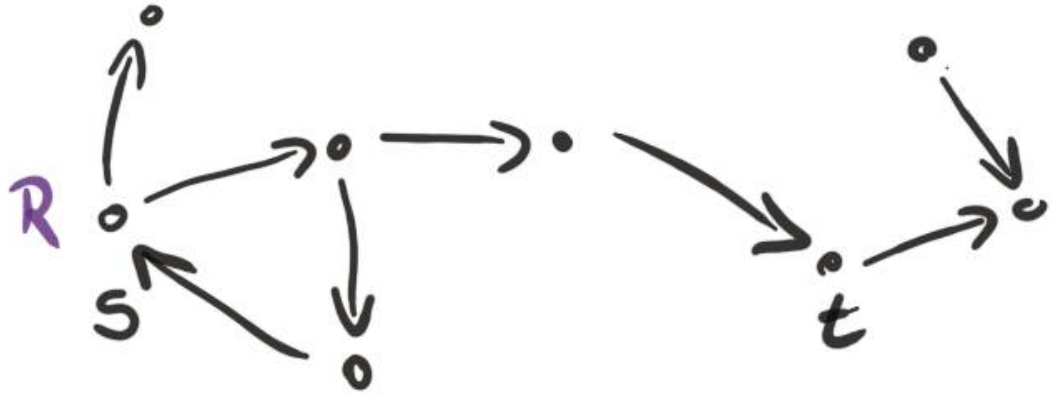
Directed Reachability and Datalog



Π_{DR}

$$R(x) :- S(x) \mid \exists y\, R(y) \wedge (y \to x)$$
$$goal :- R(x) \wedge t(x)$$

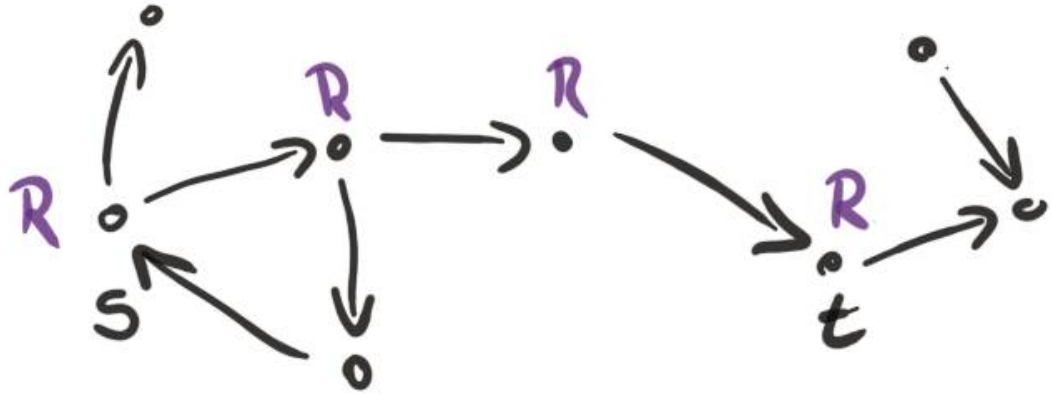
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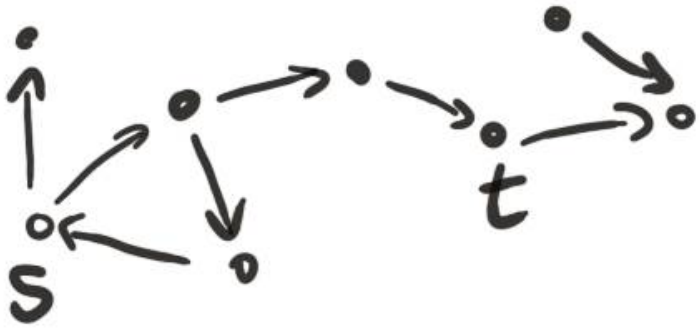


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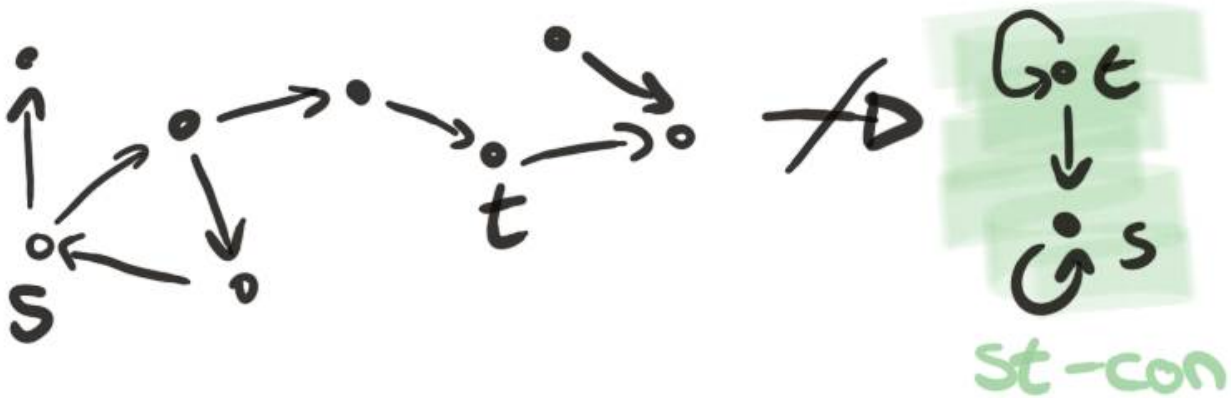
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Directed Reachability and Constraint Satisfaction Problems

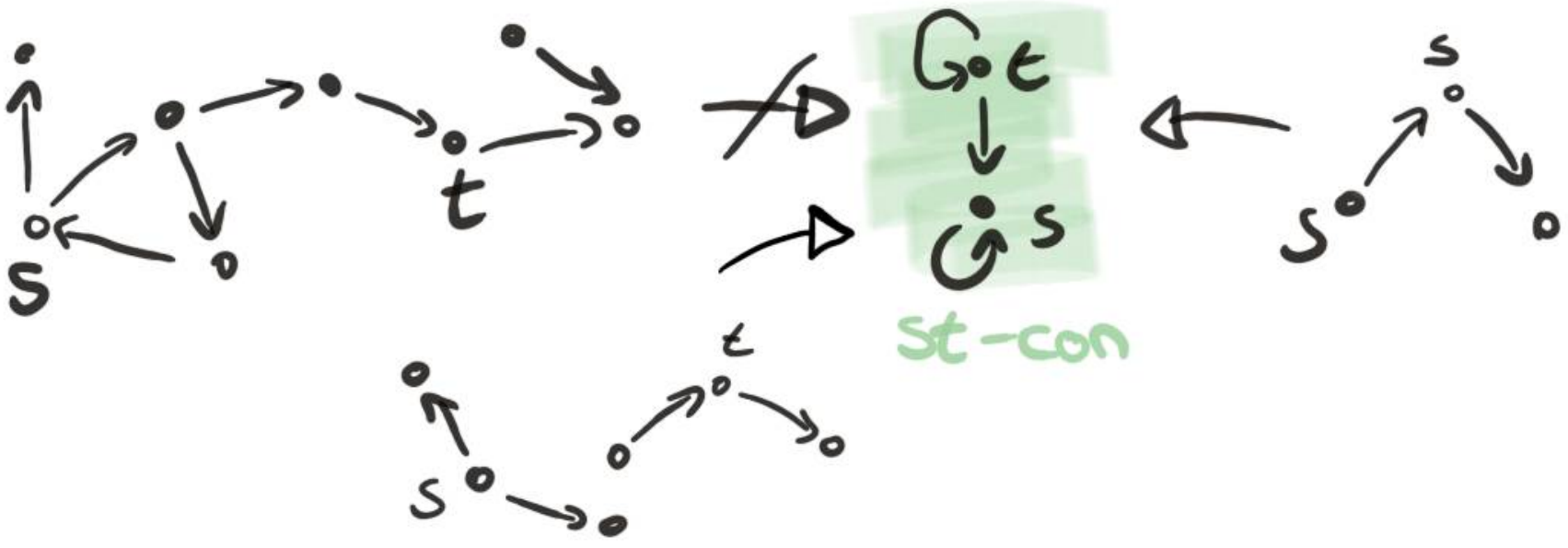
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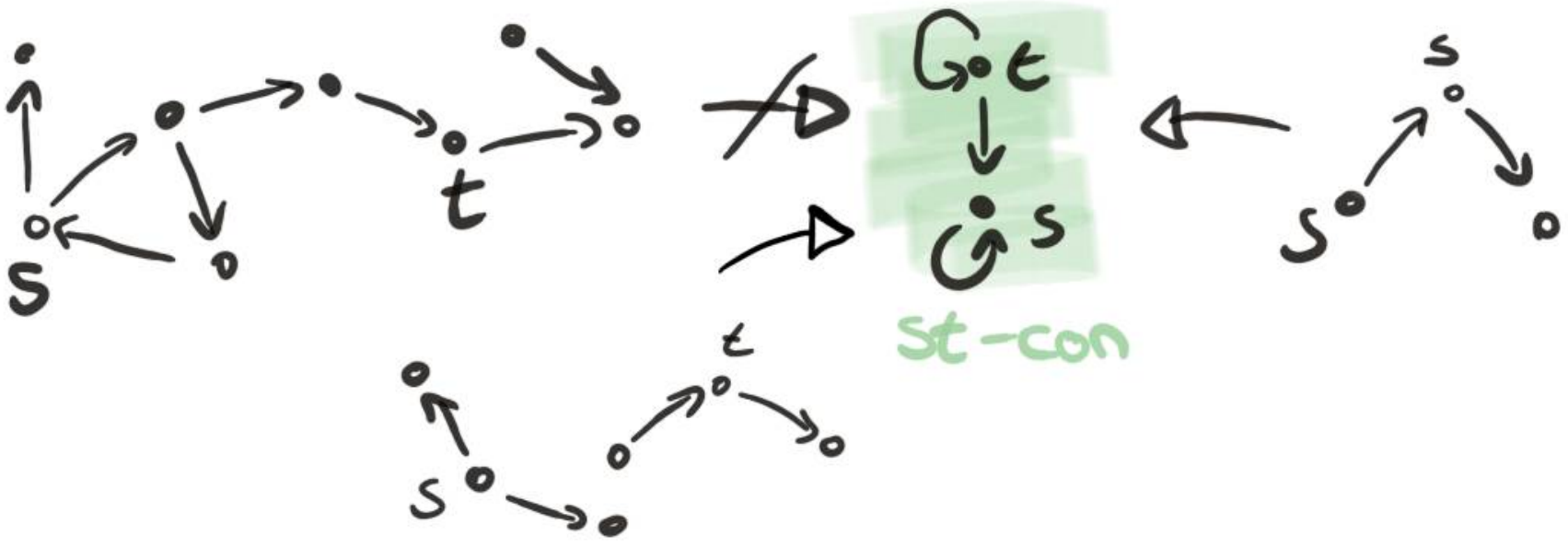
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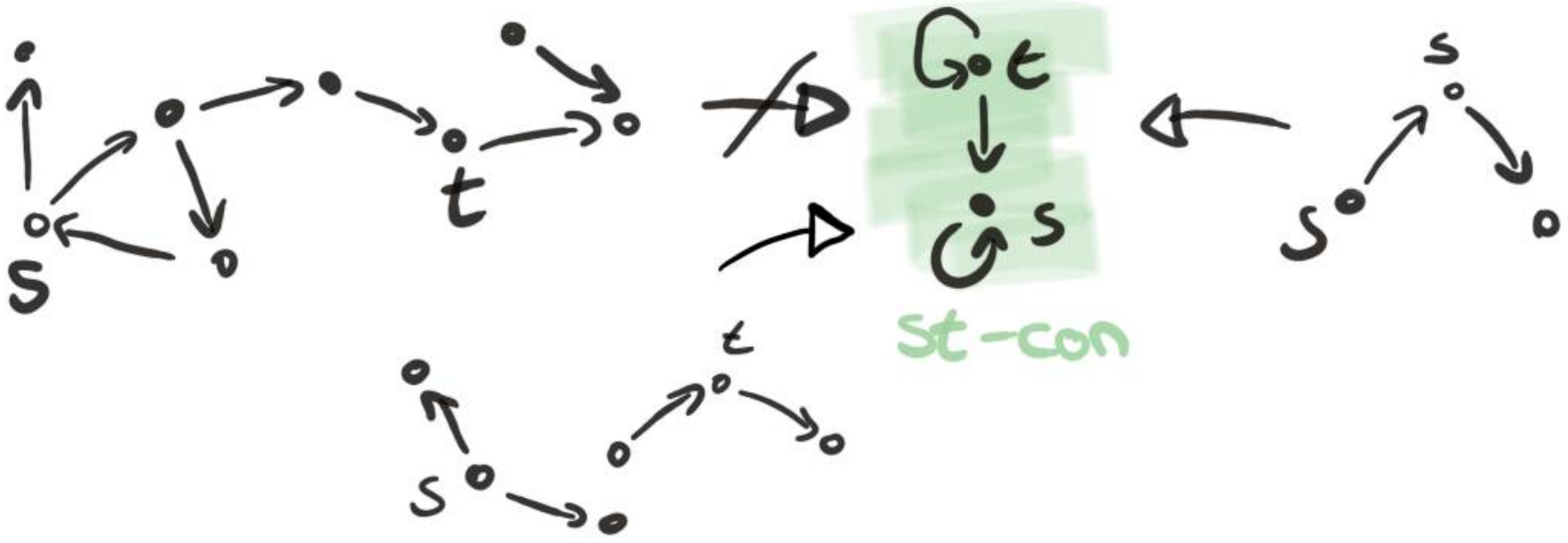


Directed Reachability and Constraint Satisfaction Problems



$$CSP(\text{st-con}) = \{ G \mid G \text{ finite}, G \to \text{st-con} \}$$

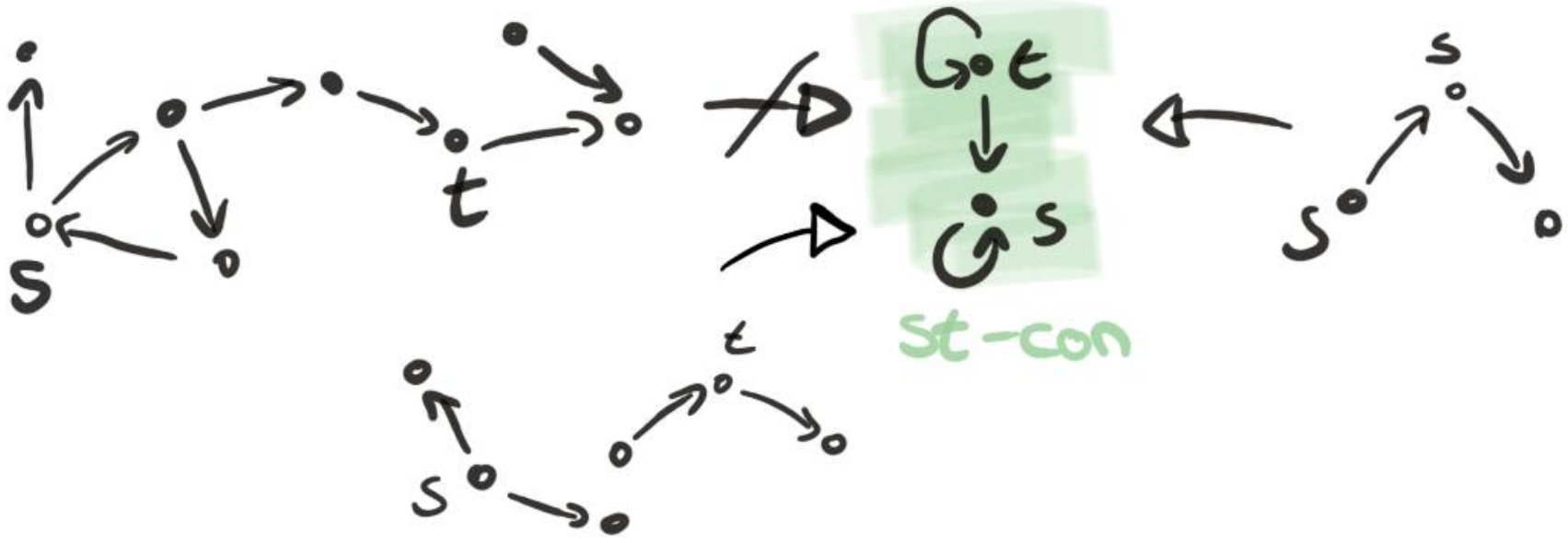
Directed Reachability and Constraint Satisfaction Problems



$$CSP(\text{st-con}) = \{G \mid G \text{ finite}, G \to \text{st-con}\}$$

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$$= \{G \mid G \text{ finite, } \Pi_{DR} \text{ does } \underline{\text{not}} \text{ derive Goal on } G\}$$

Characterisations of CSPs in Datalog

Characterisations of CSPs in Datalog

Theorem

The following are equivalent

1) $\mathrm{CSP}(\mathbb{B})$ is solved by a Datalog program
2) $\mathrm{Pol}(\mathbb{B})$ contains a 3-4 WNU
3) $\mathbb{B}$ has a bounded tree width duality

Characterisations of CSPs in Datalog Fragments

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Theorem

The following are equivalent

1) CSP(IB) is solved by an AM Datalog program
2) Pol(IB) contains totally symmetric polymorpisms of all arities
3) IB has tree duality
4) Horn SAT $\leq_{pp}$ IB

Characterisations of CSPs in Datalog Fragments

Theorem

The following are equivalent

1) CSP(IB) is solved by an AM Datalog program (at most one EDB in the Body; IDBs are unary; are consistency)
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$(\{0,1\}, x \wedge y \Rightarrow z, \{0\}, \{1\})$



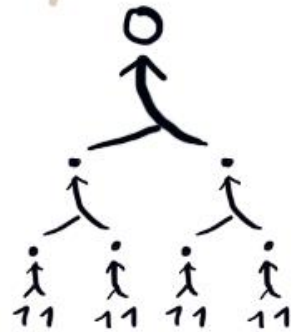
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Characterisations of CSPs in Datalog Fragments

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Theorem (Carvalho, Dalmau, Krokhin in 2011)

The following are equivalent

1) CSP(IB) is solved by a LAM Datalog program
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Characterisations of CSPs in Datalog Fragments

Theorem (Carvalho, Dalmau, Krokhin in 2011)

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$s \to \cdot \to \cdot \to \cdot \to t$





Datalog Fragments

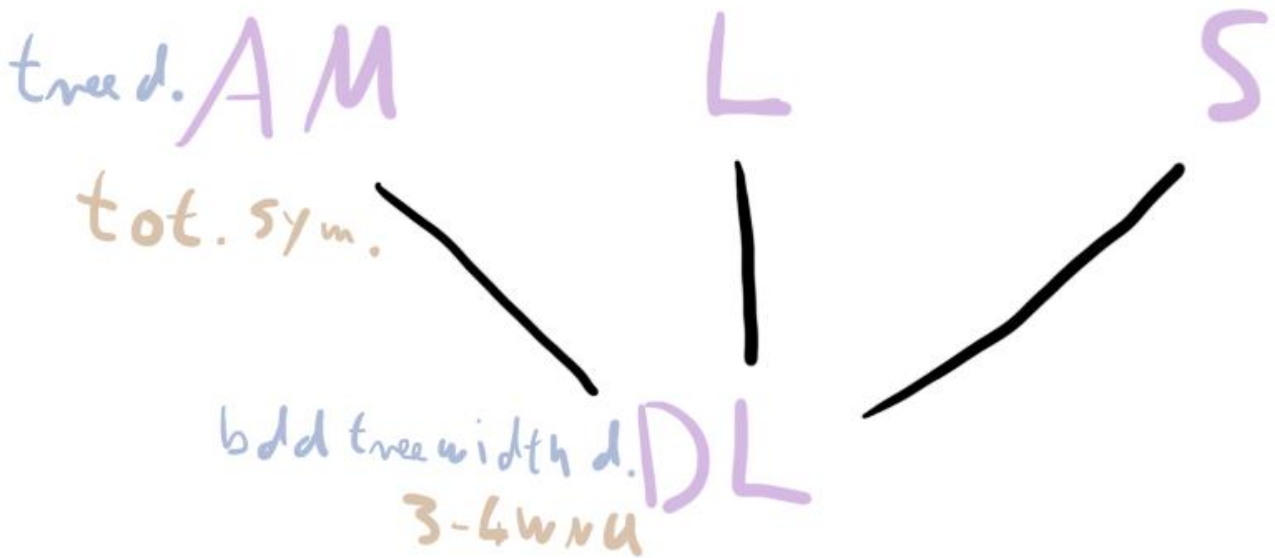
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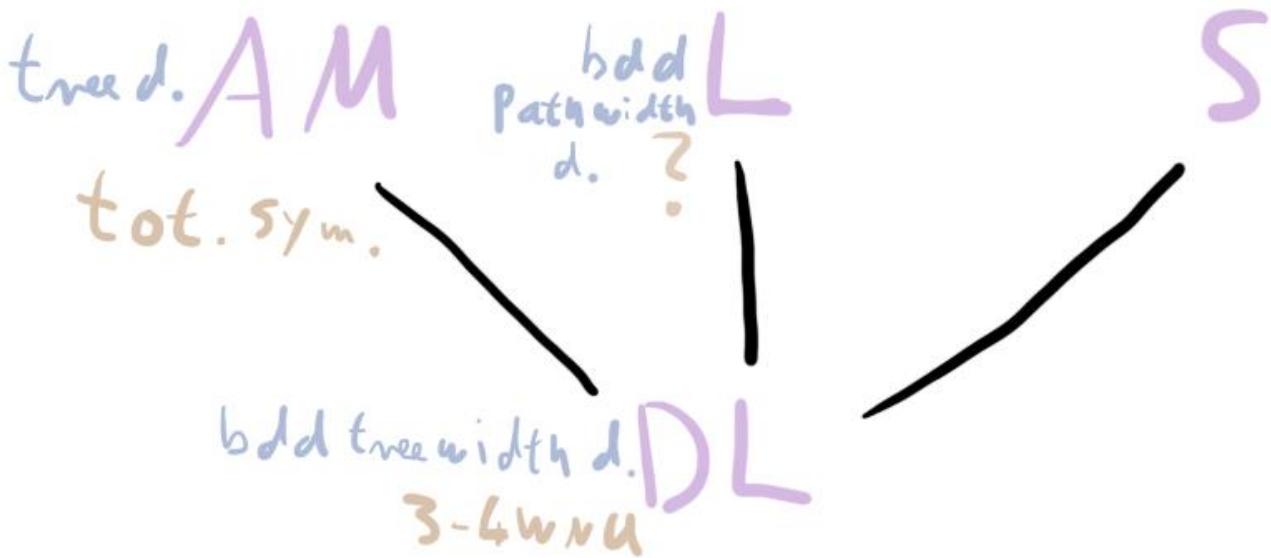
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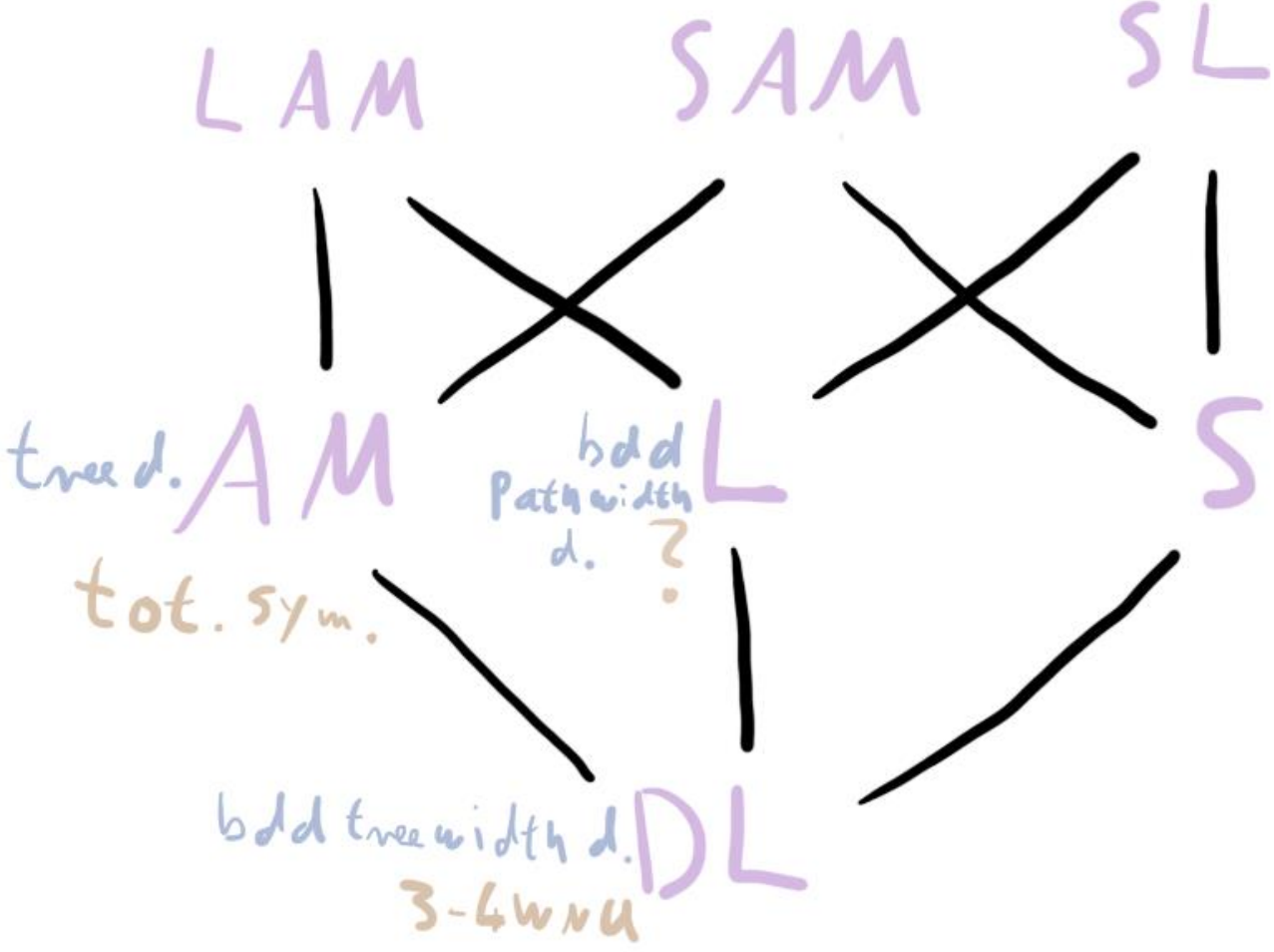
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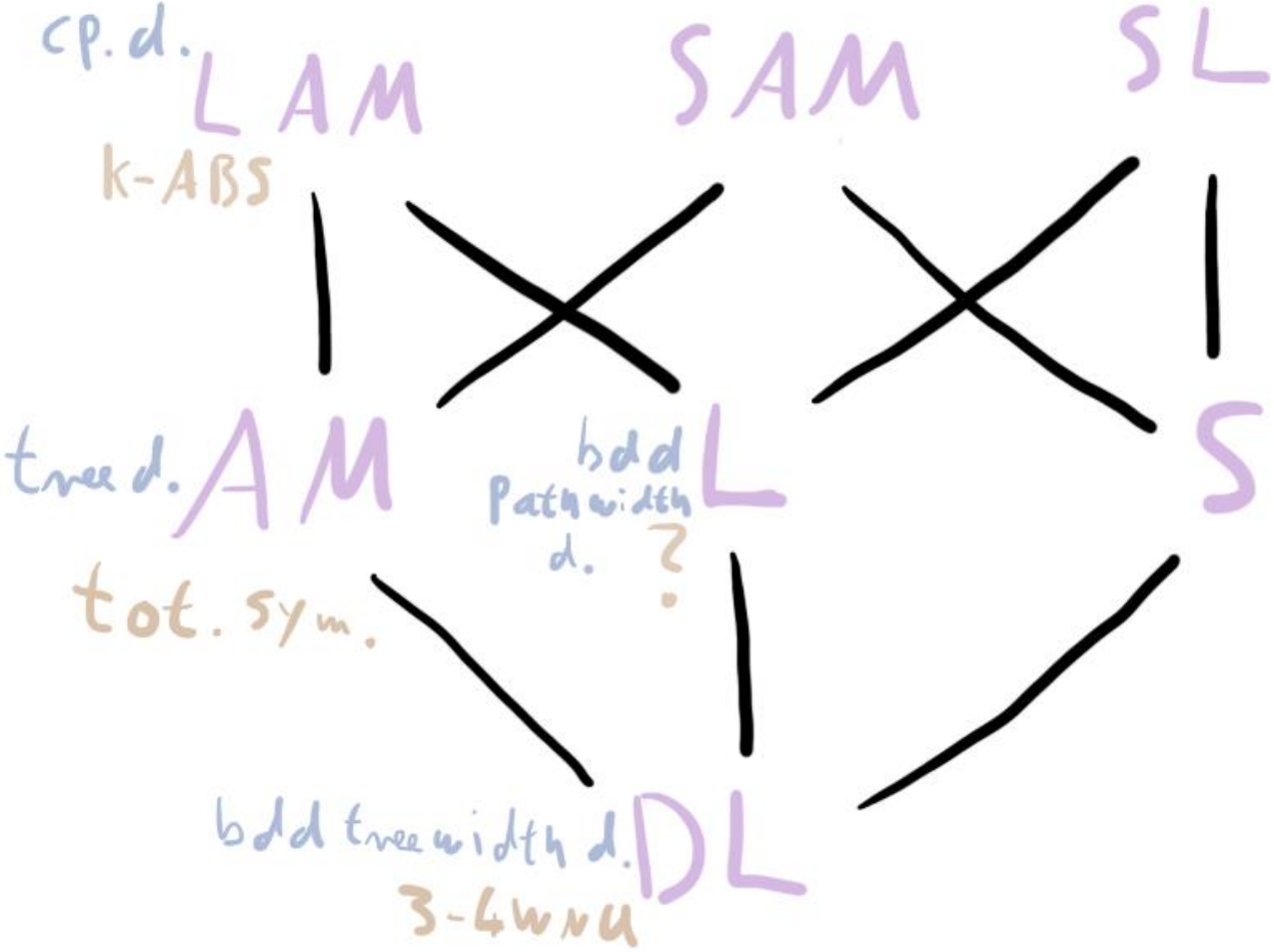
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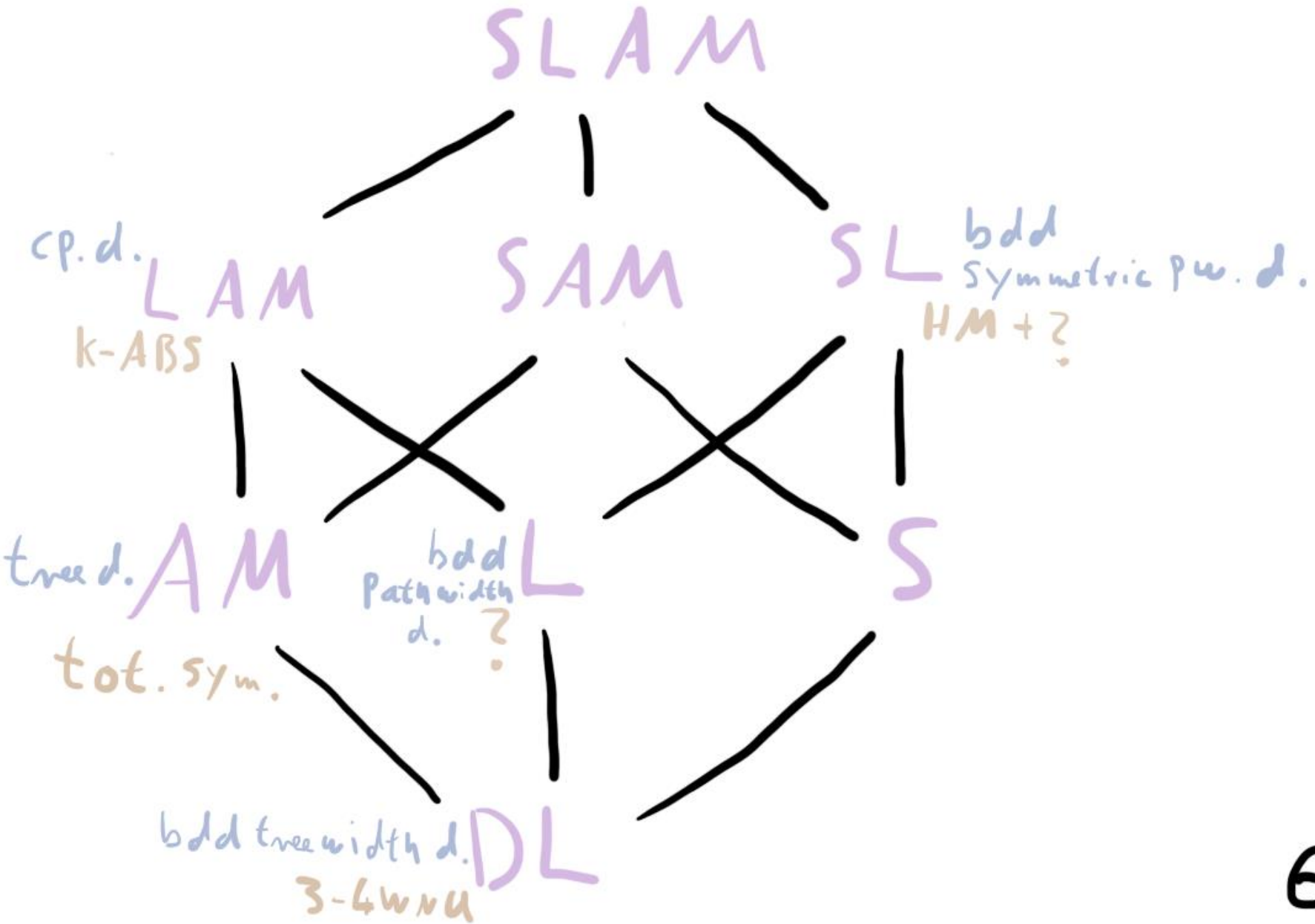
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Characterisations of CSPs in Datalog Fragments

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Theorem (Bodirsky, Starke in 2025)

The following are equivalent

1) $\mathrm{CSP}(\mathbb{B})$ is solved by a SLAM Datalog program
2) $\mathrm{Pol}(\mathbb{B})$ contains K-ABS polymorphisms of all arities and Ele(1)
3) $\mathbb{B}$ has unfolded caterpillar duality
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Characterisations of CSPs in Datalog Fragments

Theorem (Bodirsky, Starke in 2025)

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4) $T_2 \leq_{pp} \mathbb{B}$

Characterisations of CSPs in Datalog Fragments

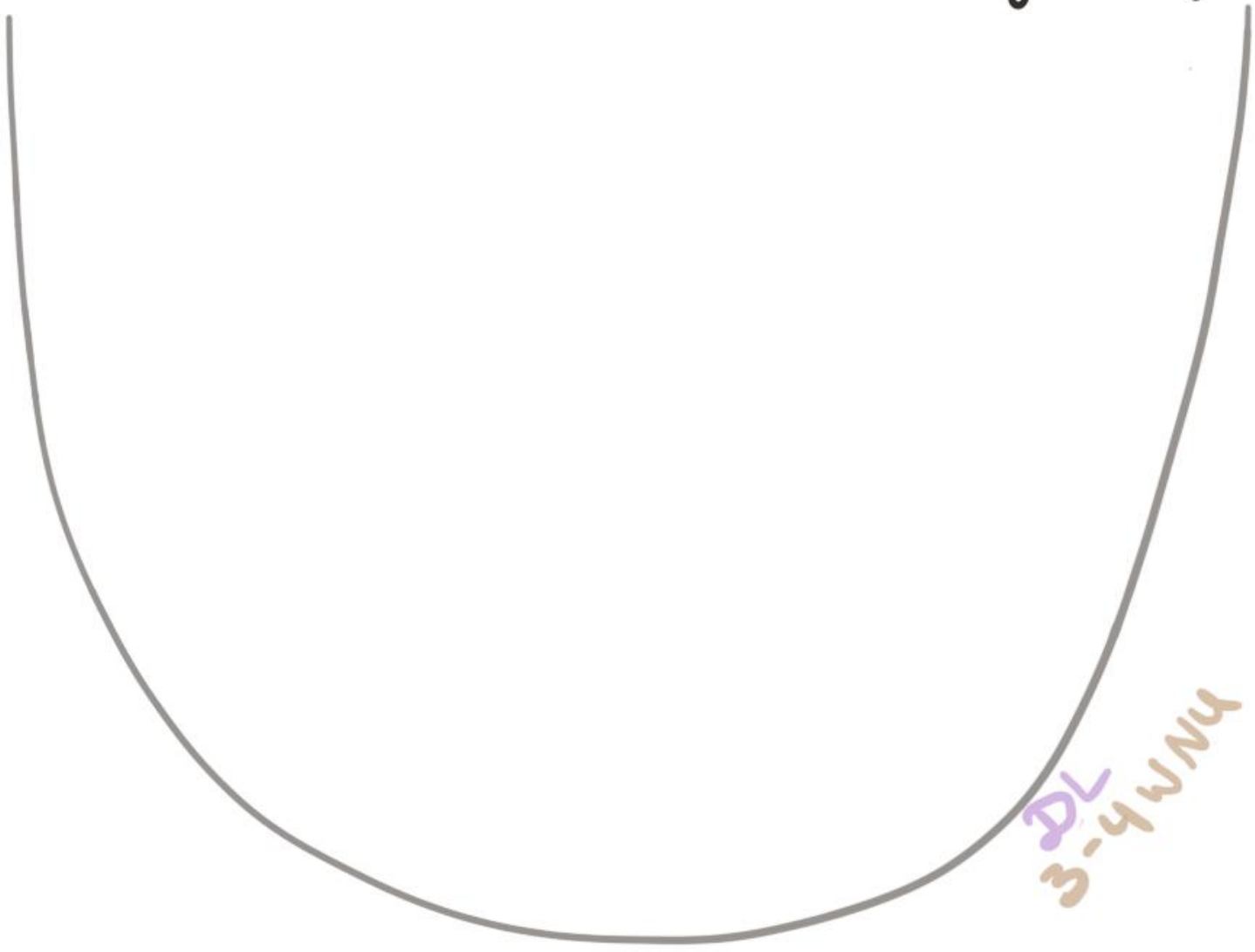
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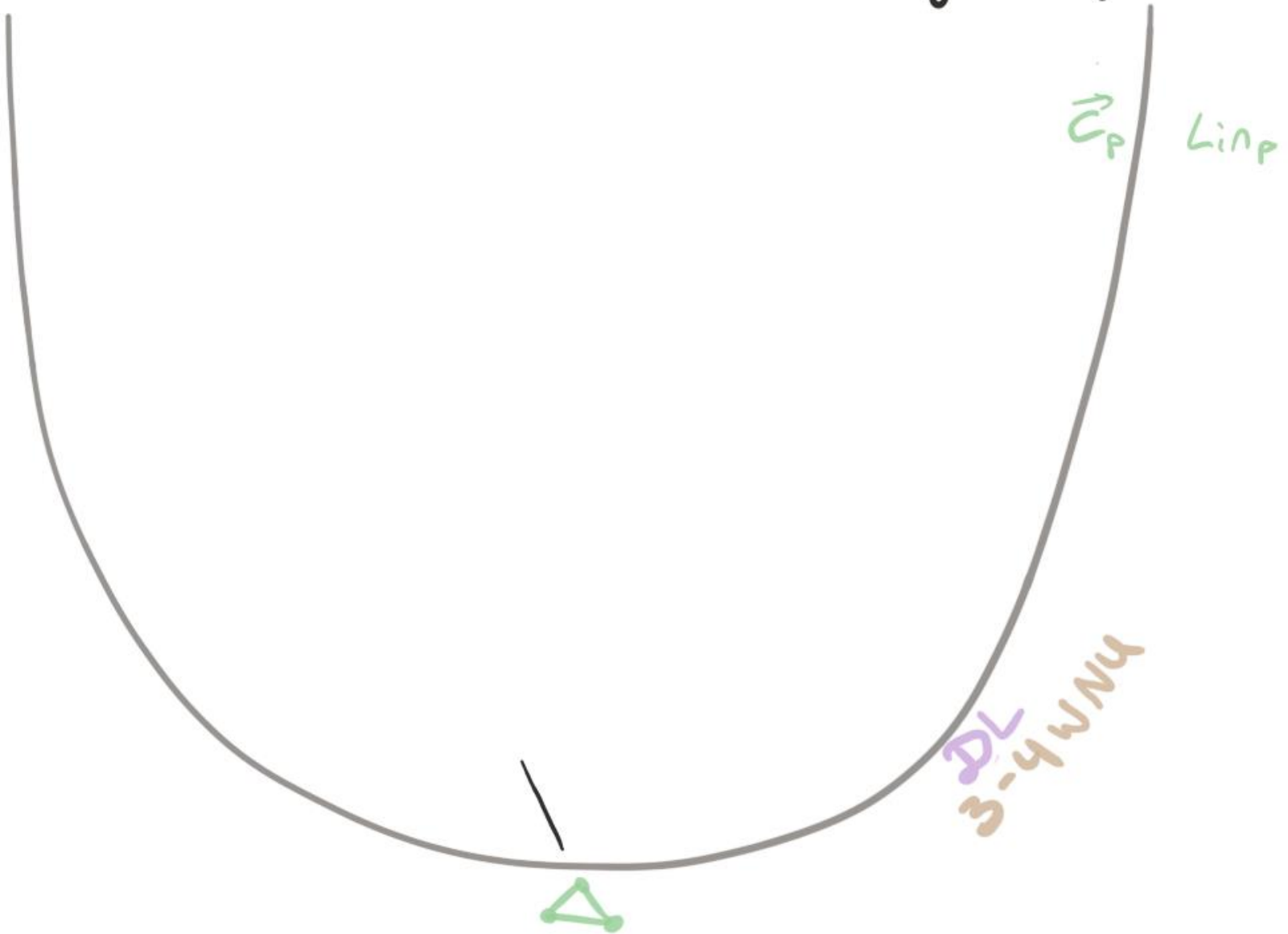
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↑ ! ⇏ 𝔹 implies !N! ⇏ 𝔹

Minor Conditions & Datalog Fragments



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Minor Conditions & Datalog Fragments



CSPs of Transitive Tournaments

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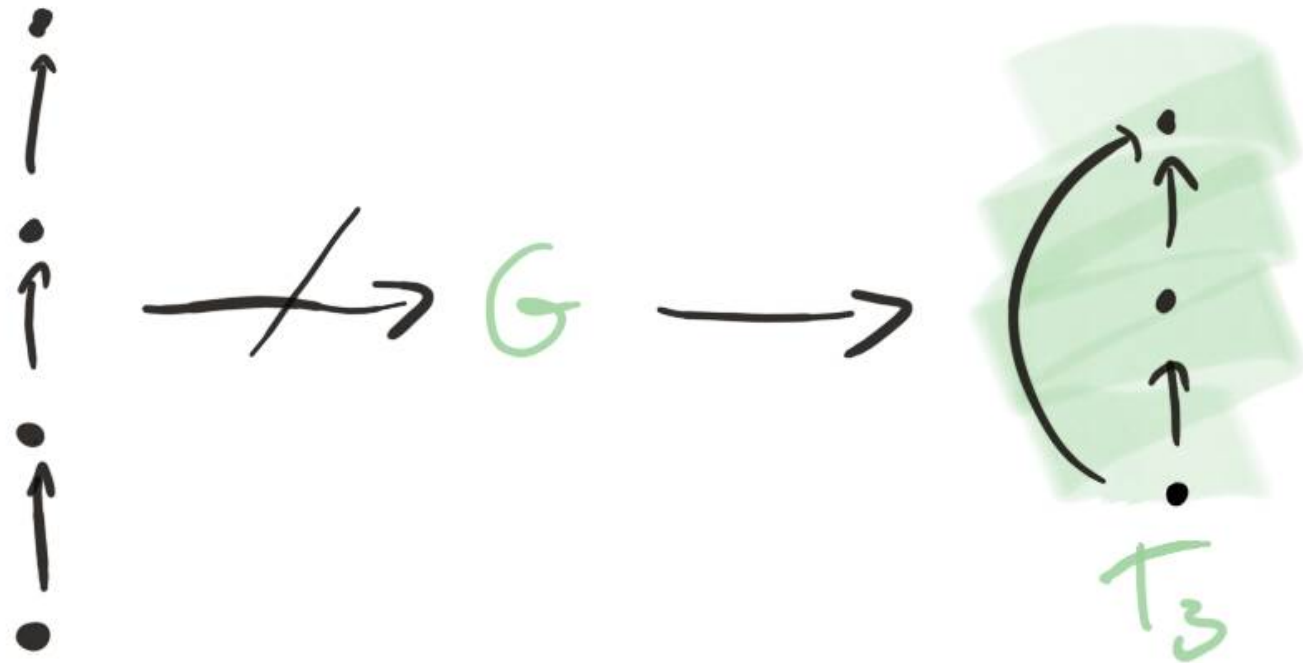


CSPs of Transitive Tournaments

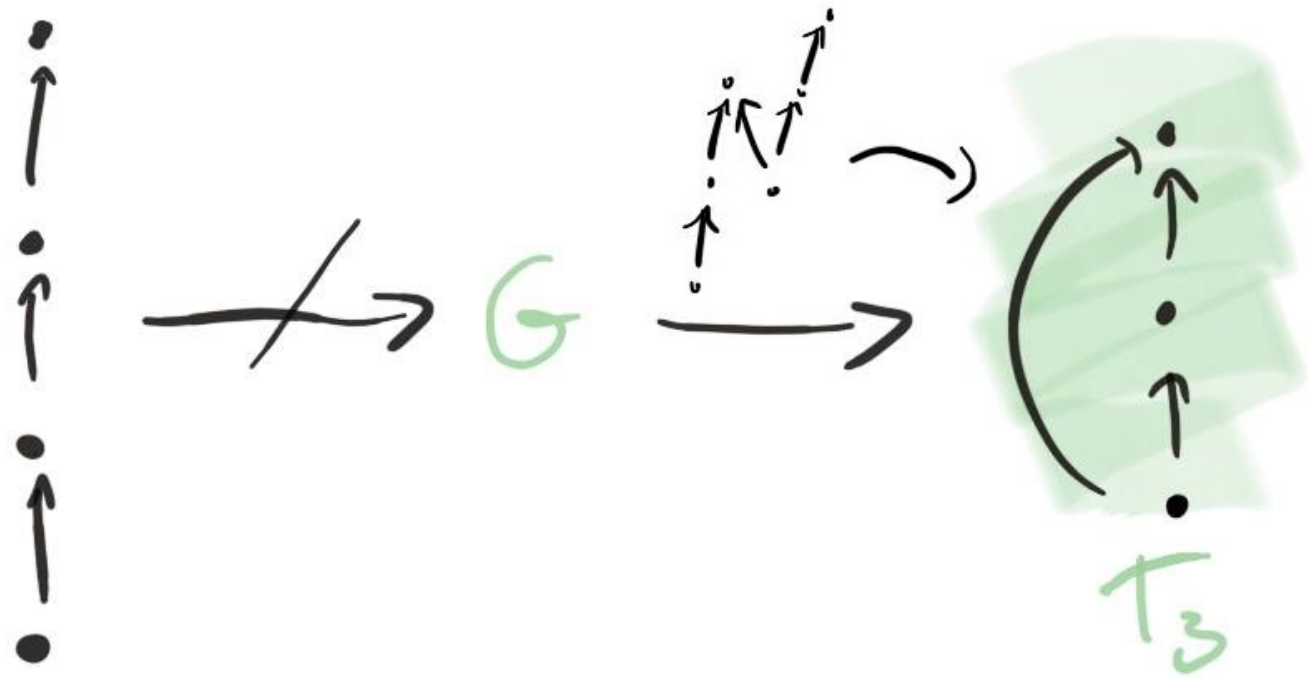




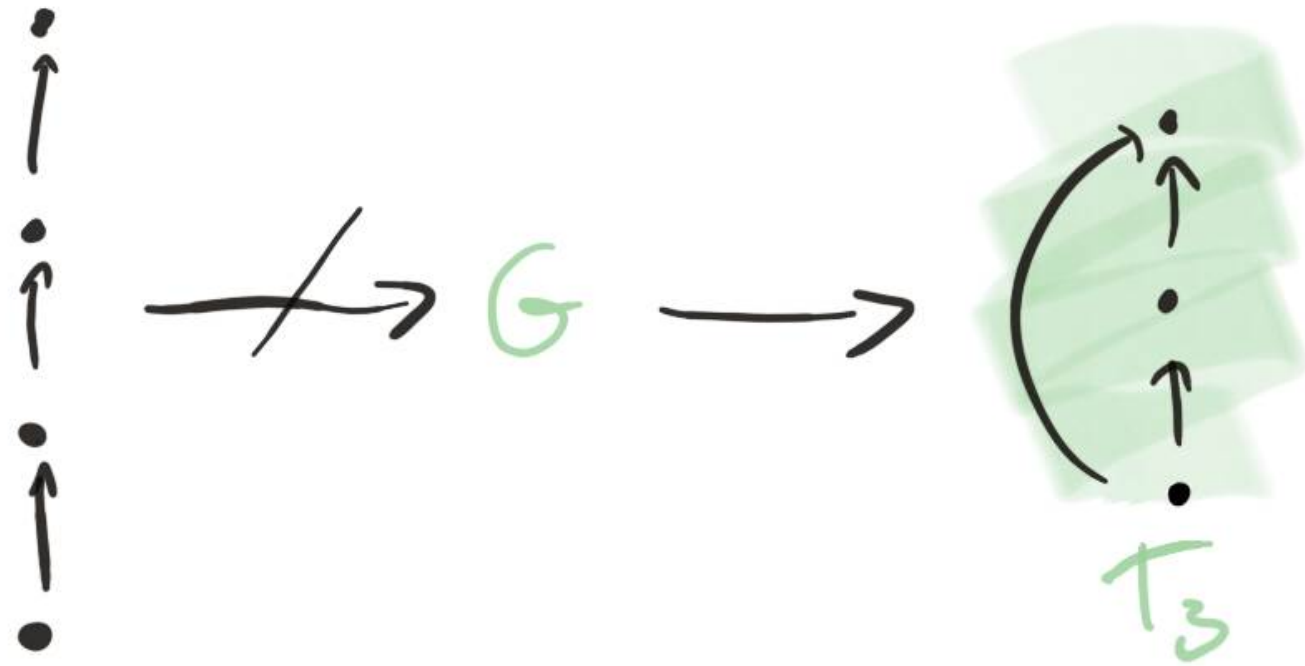
CSPs of Transitive Tournaments



CSPs of Transitive Tournaments



CSPs of Transitive Tournaments



Goal: $\neg\exists x_0\, x_1\, x_2\, x_3 \qquad x_0 \to x_1 \to x_2 \to x_3$

CSPs of Transitive Tournaments

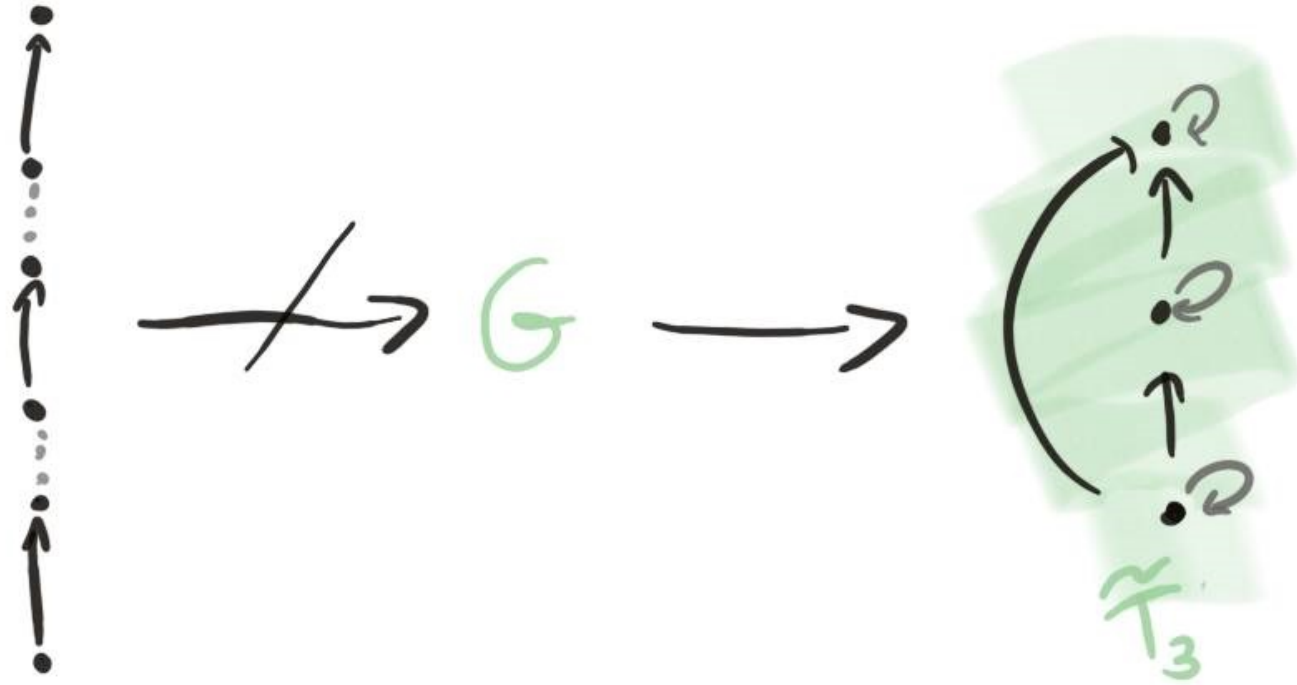


CSPs of Transitive Tournaments

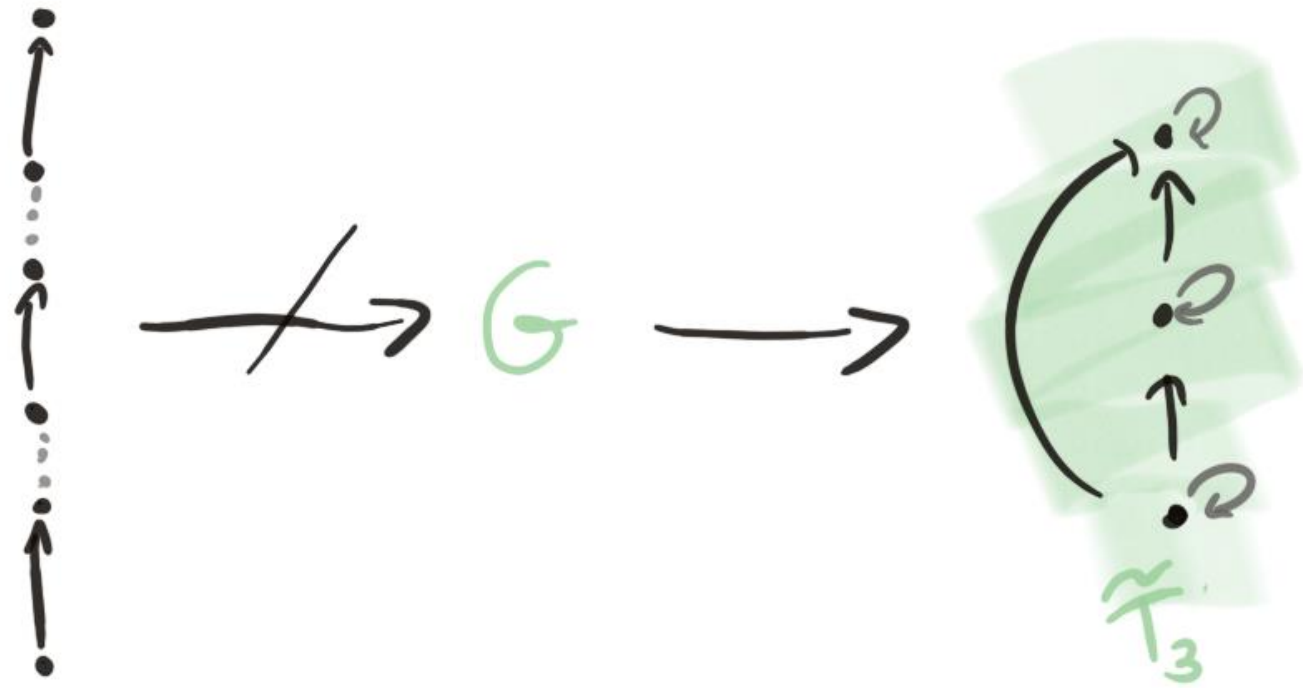




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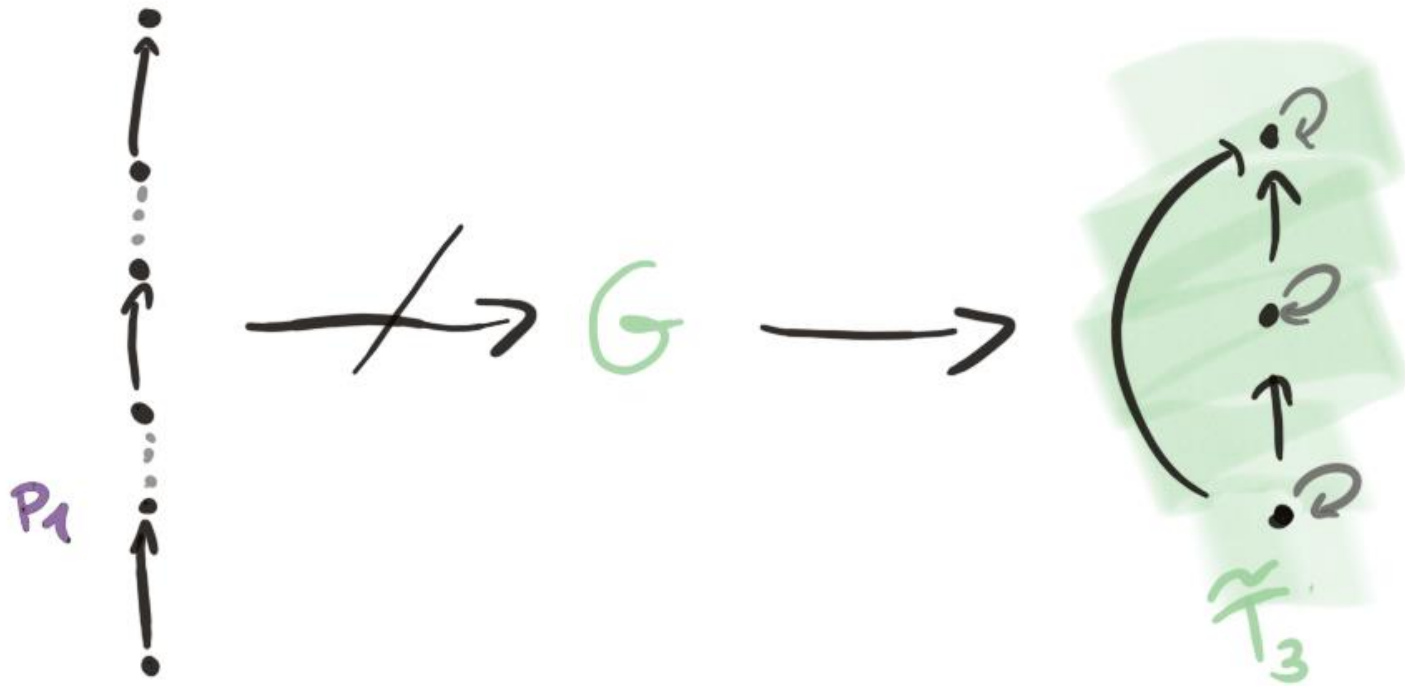


$P_1(x) :- \exists y \; y \to x$

$P_2(x) :- \exists y \; P_1(y) \wedge y \to x$

$\text{Goal} :- \exists x, y \; P_2(y) \wedge y \to x$

CSPs of Transitive Tournaments

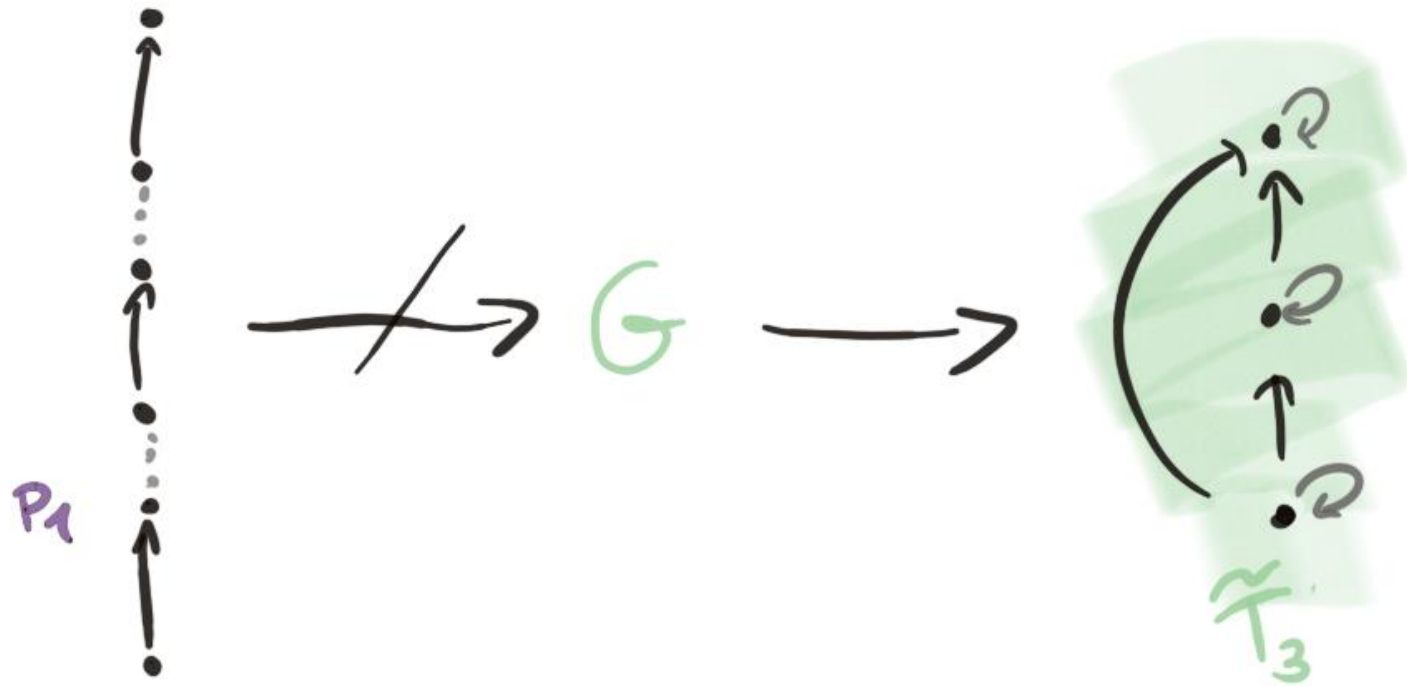


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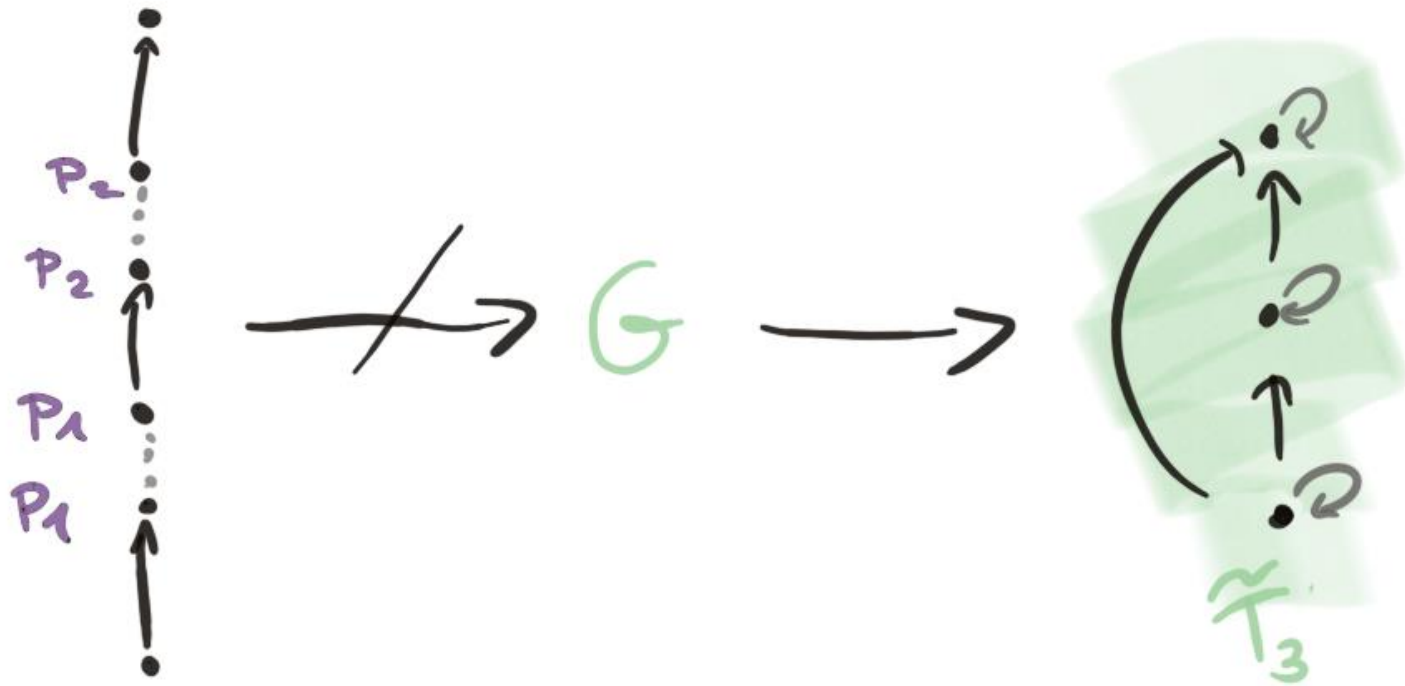


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n-almost symmetric linear arc monadic Datalog

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Definition

Π is a S_n LAM DL program if

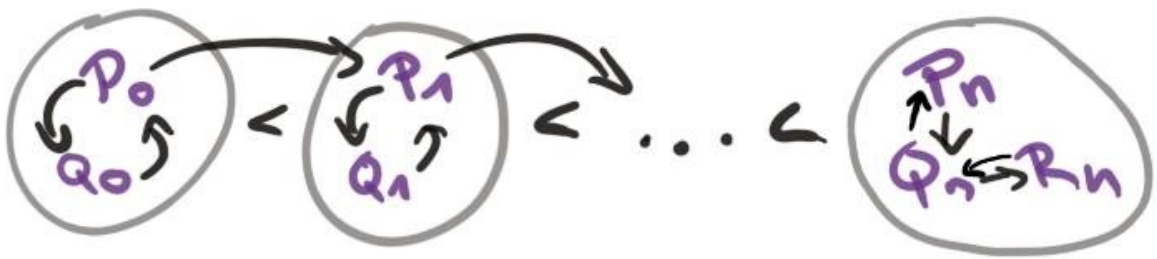
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n-almost symmetric linear arc monadic Datalog

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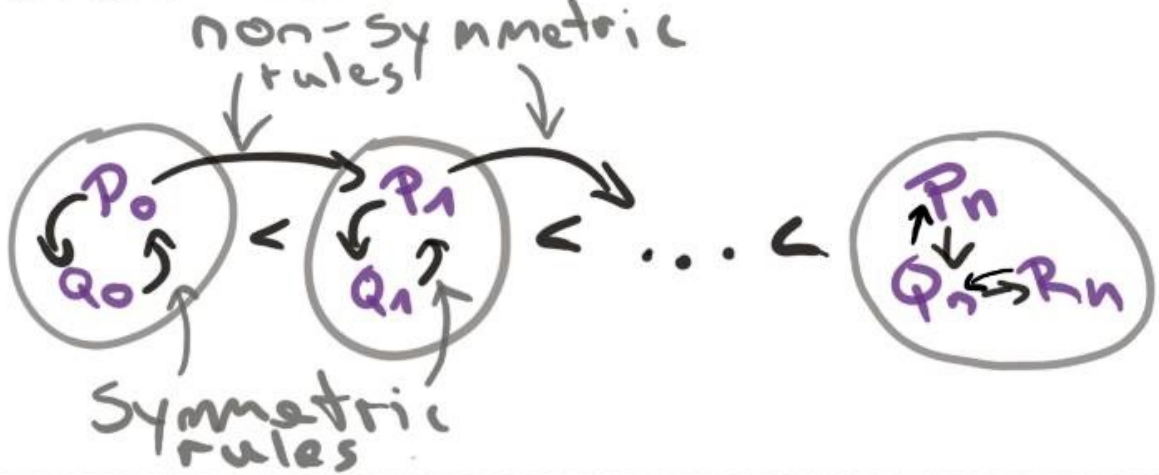


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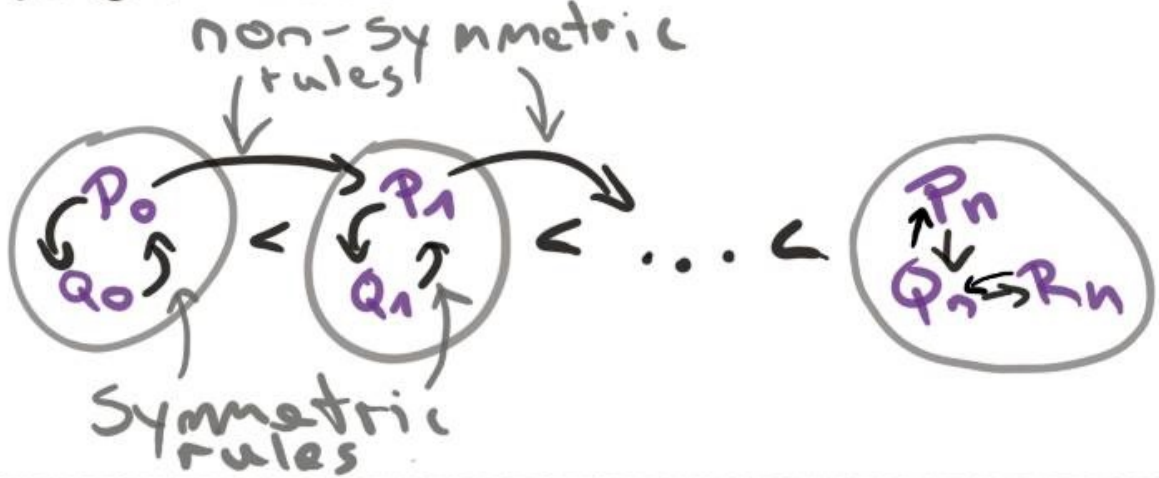


n-almost symmetric linear are monadic Datalog

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every derivation of Π applies non-symmetric rules at most n times

Almost symmetric linear Datalog

Almost symmetric linear Datalog



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Almost symmetric linear Datalog



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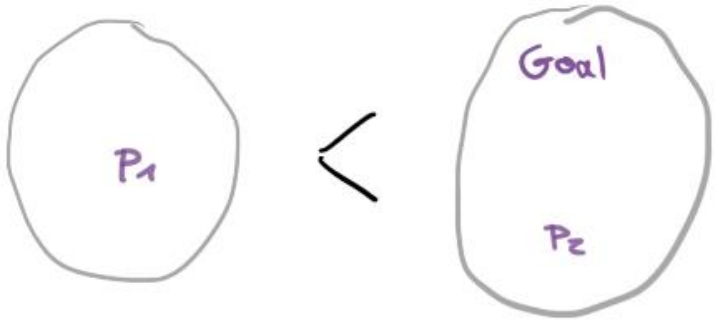
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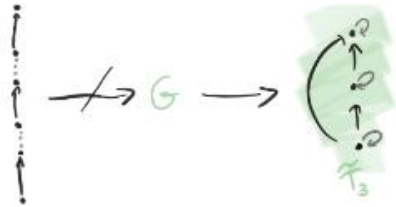
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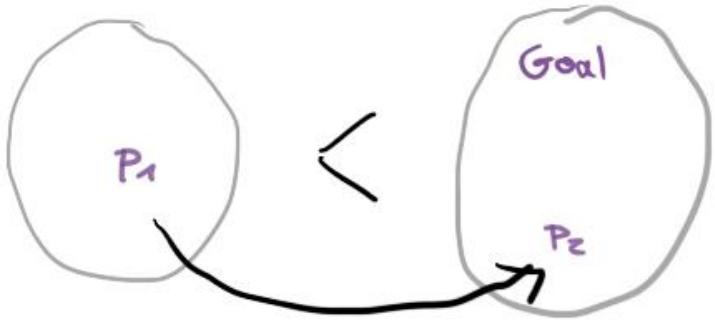
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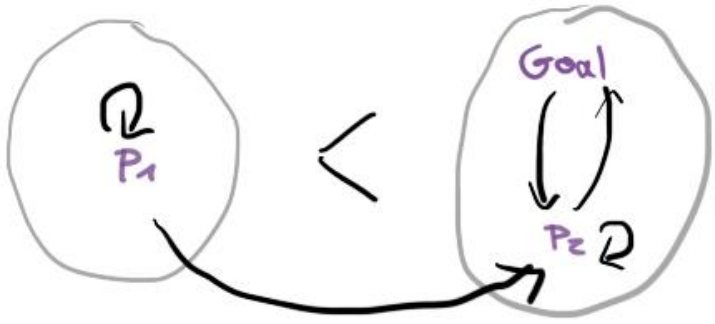
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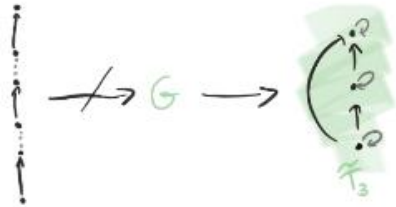
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$P_2(x) :- \exists y\ P_1(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \leftarrow x$

$Goal :- \exists x, y\ P_2(y) \wedge y \to x \qquad P_2(y) :- \exists x\ Goal \wedge y \to x$



Almost symmetric linear Datalog



$P_1(x) :- \exists y\ y \to x \mid \exists y\ P_1(y) \wedge y \to x \mid \exists y\ P_1(y) \wedge y \leftarrow x$

$P_2(x) :- \exists y\ P_1(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \leftarrow x$

$Goal :- \exists x, y\ P_2(y) \wedge y \to x \quad P_2(y) :- \exists x\ Goal \wedge y \to x$

$$P(x_1', x_1, x_2', x_2) :- \exists y, z\ \ y \to x_1' \wedge x_1 \to x_2' \wedge x_2 \to z \mid$$

$$\exists x_1''\ \ P(x_1'', x_1, x_2', x_2) \wedge x_1'' \to x_1' \mid$$

$$\exists x_1''\ \ P(x_1'', x_1, x_2', x_2) \wedge x_1'' \leftarrow x_1' \mid$$

$$\exists x_2''\ \ P(x_1', x_1, x_2'', x_2) \wedge x_2'' \to x_2' \mid$$

$$\exists x_2''\ \ P(x_1', x_1, x_2'', x_2) \wedge x_2' \leftarrow x_2'$$

$$Goal :- P(x_1, x_1, x_2, x_2)$$

Almost symmetric linear Datalog



$P_1(x) :- \exists y\ y \to x \mid \exists y\ P_1(y) \wedge y \to x \mid \exists y\ P_1(y) \wedge y \leftarrow x$

$P_2(x) :- \exists y\ P_1(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \to x \mid \exists y\ P_2(y) \wedge y \leftarrow x$

$Goal :- \exists x,y\ P_2(y) \wedge y \to x \quad P_2(y) :- \exists x\ Goal \wedge y \to x$

$$P(x_1', x_1, x_2', x_2) :- \exists y, z\ \ y \to x_1' \wedge x_1 \to x_2' \wedge x_2 \to z \mid$$
$$\exists x_1''\ \ P(x_1'', x_1, x_2', x_2) \wedge x_1'' \to x_1' \mid$$
$$\exists x_1''\ \ P(x_1'', x_1, x_2', x_2) \wedge x_1'' \leftarrow x_1' \mid$$
$$\exists x_2''\ \ P(x_1', x_1, x_2'', x_2) \wedge x_2'' \to x_2' \mid$$
$$\exists x_2''\ \ P(x_1', x_1, x_2'', x_2) \wedge x_2' \leftarrow x_2'$$
$$Goal :- P(x_1, x_1, x_2, x_2)$$

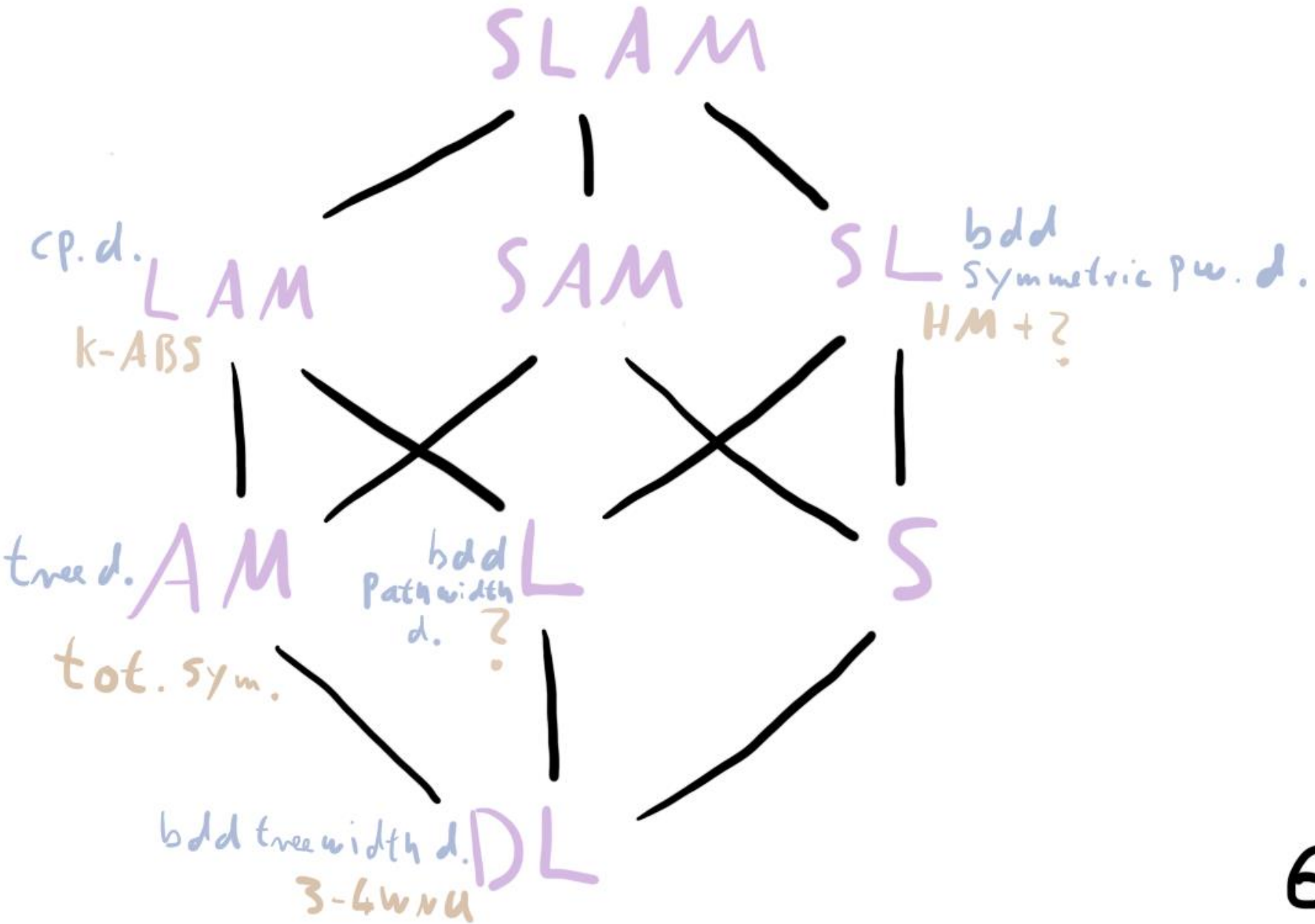
Lemma

CSP($\mathbb{B}$) is solved by a SL Datalog program

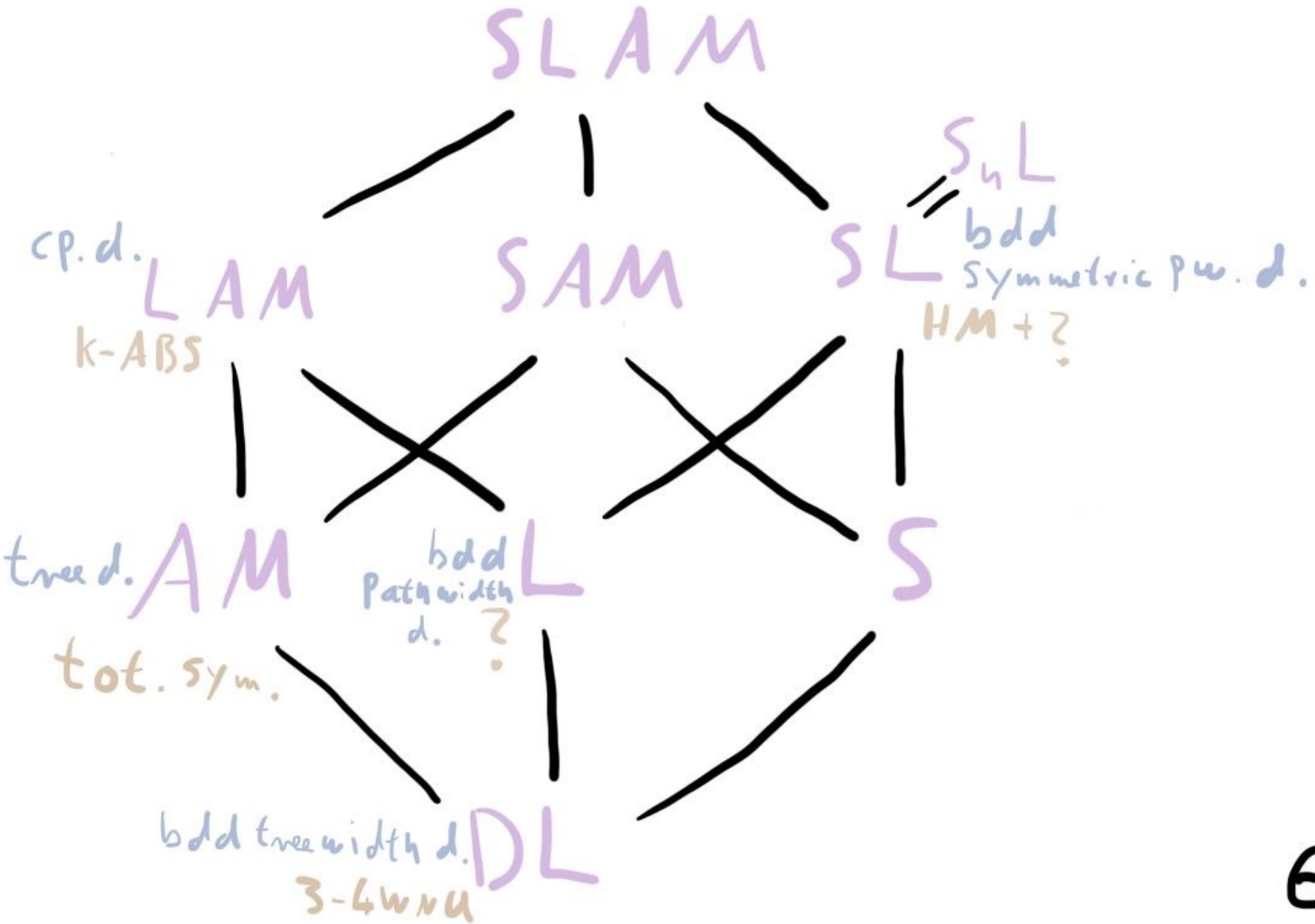
iff

CSP($\mathbb{B}$) is solved by a S_nL Datalog program

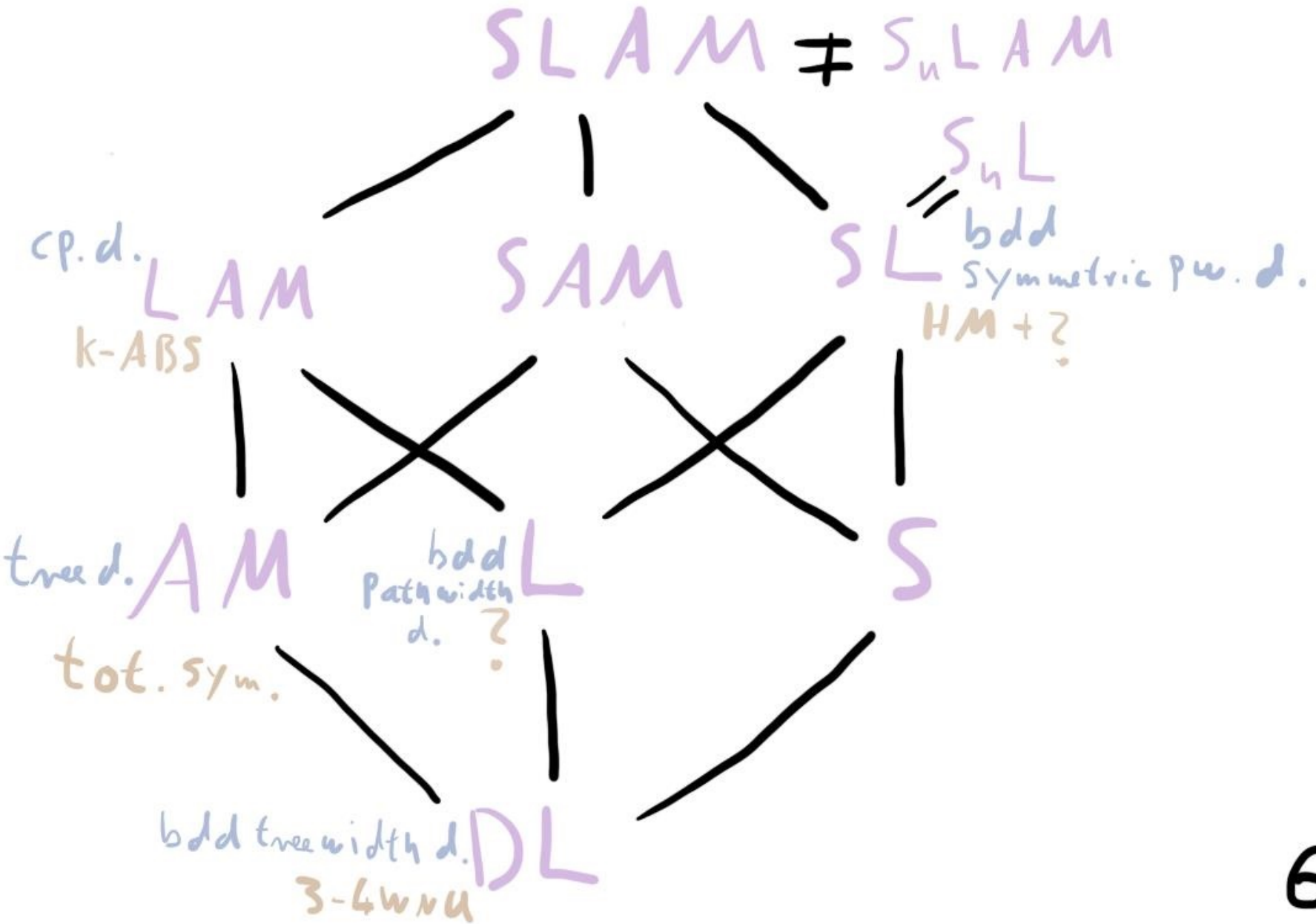
Datalog Fragments



Datalog Fragments



Datalog Fragments



Main Result

Main Result

Theorem (Meyer, starke in 2026)

The following are equivalent

1) CSP(𝔹) is solved by a SnLAM Datalog program
2) Pol(𝔹) contains K-ABS polymorpisms of all arities and Ele(n+1)
3) 𝔹 has n-fixed unfolded caterpillar duality
4) $T_{n+2} \leq_{pp} \mathbb{B}$

Main Result

Theorem (Meyer, starke in 2026)

The following are equivalent

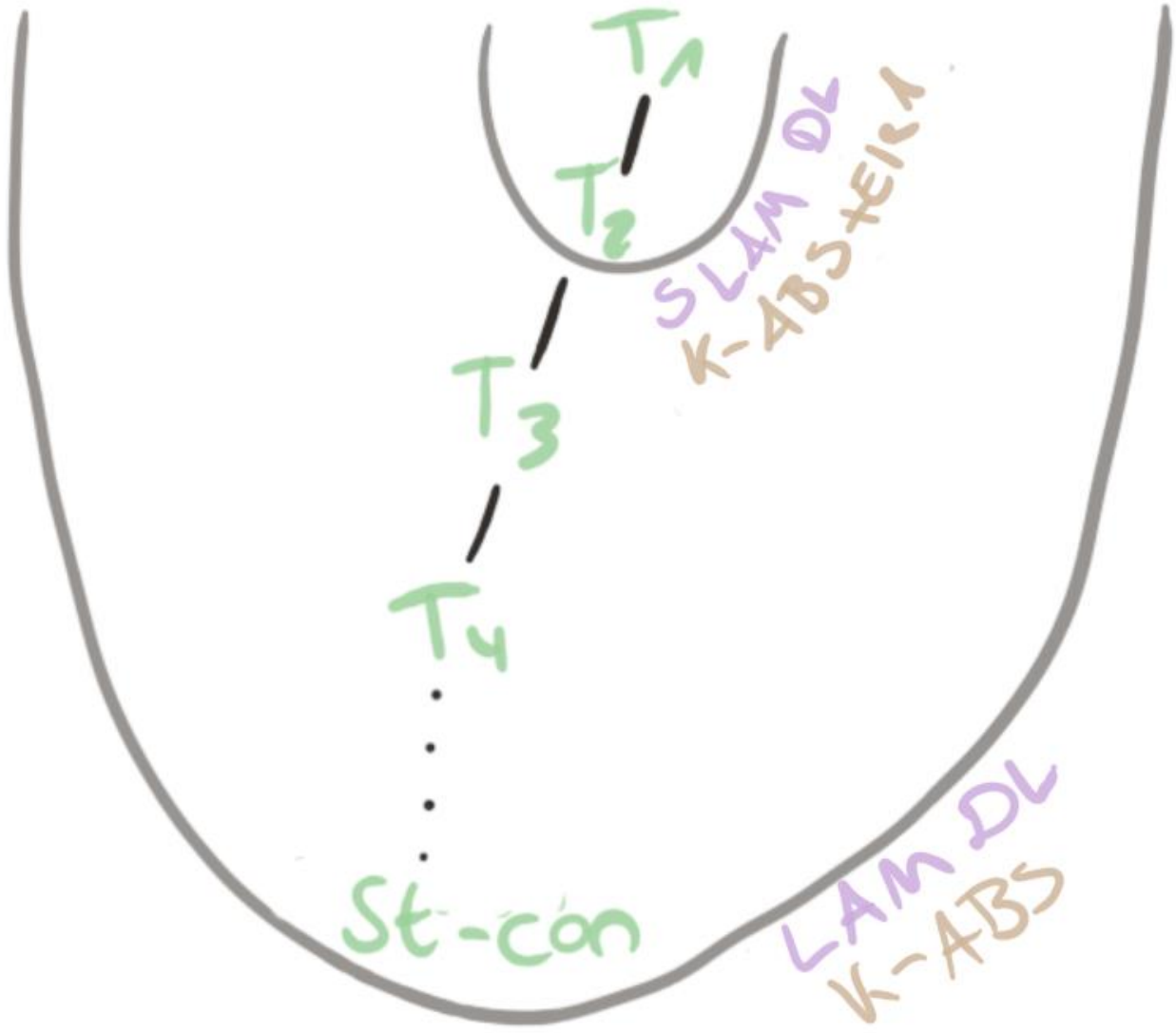
1) CSP(𝔹) is solved by a SnLAM Datalog program
2) Pol(𝔹) contains K-ABS polymorpisms of all arities and Ele(n+1)
3) 𝔹 has n-fixed unfolded caterpillar duality
4) $T_{n+2} \leq_{pp} \mathbb{B}$

↑ ⇏ 𝔹 implies ⇏ 𝔹

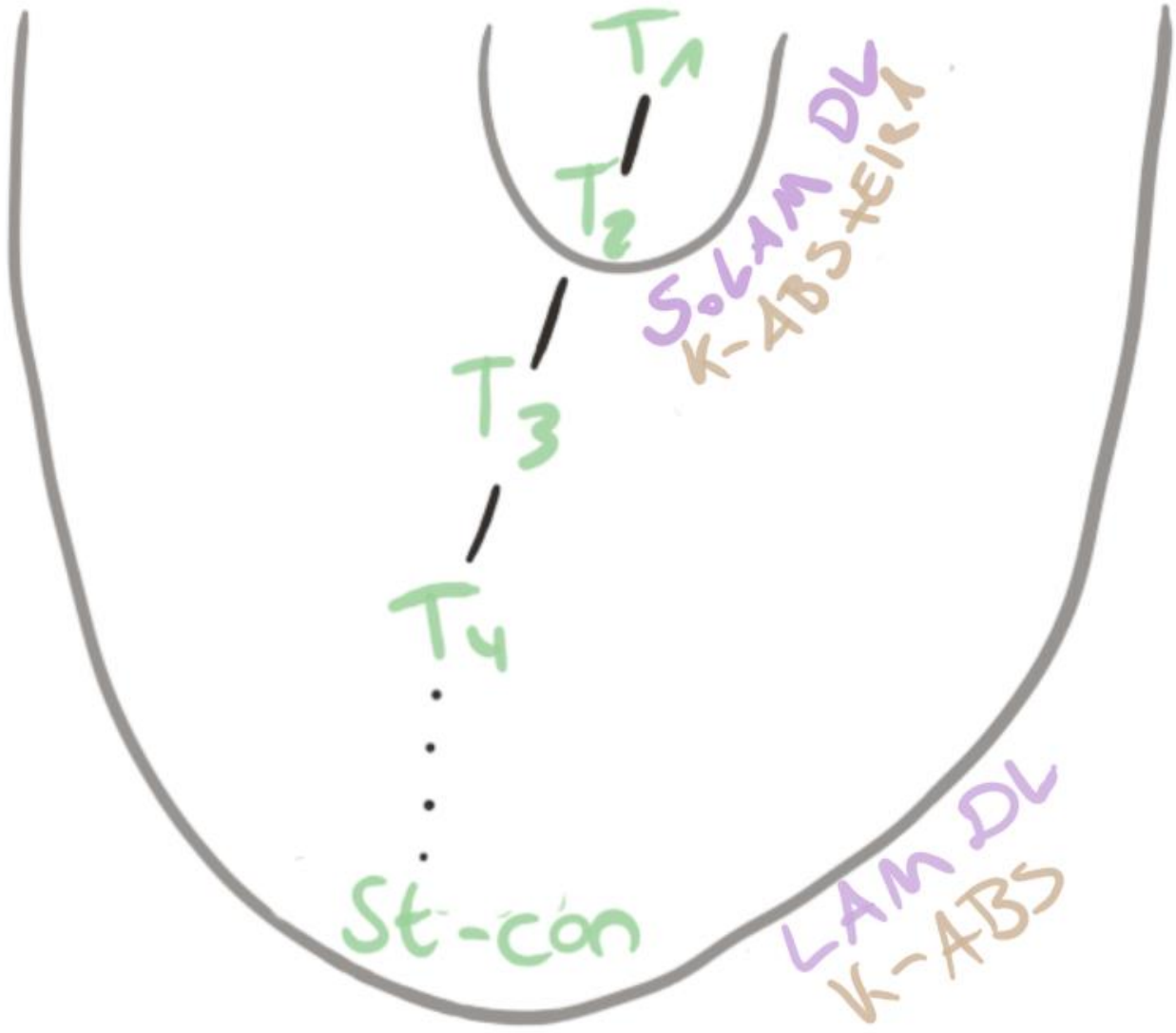
↑, ↑ can have homomorphism to 𝔹

Results

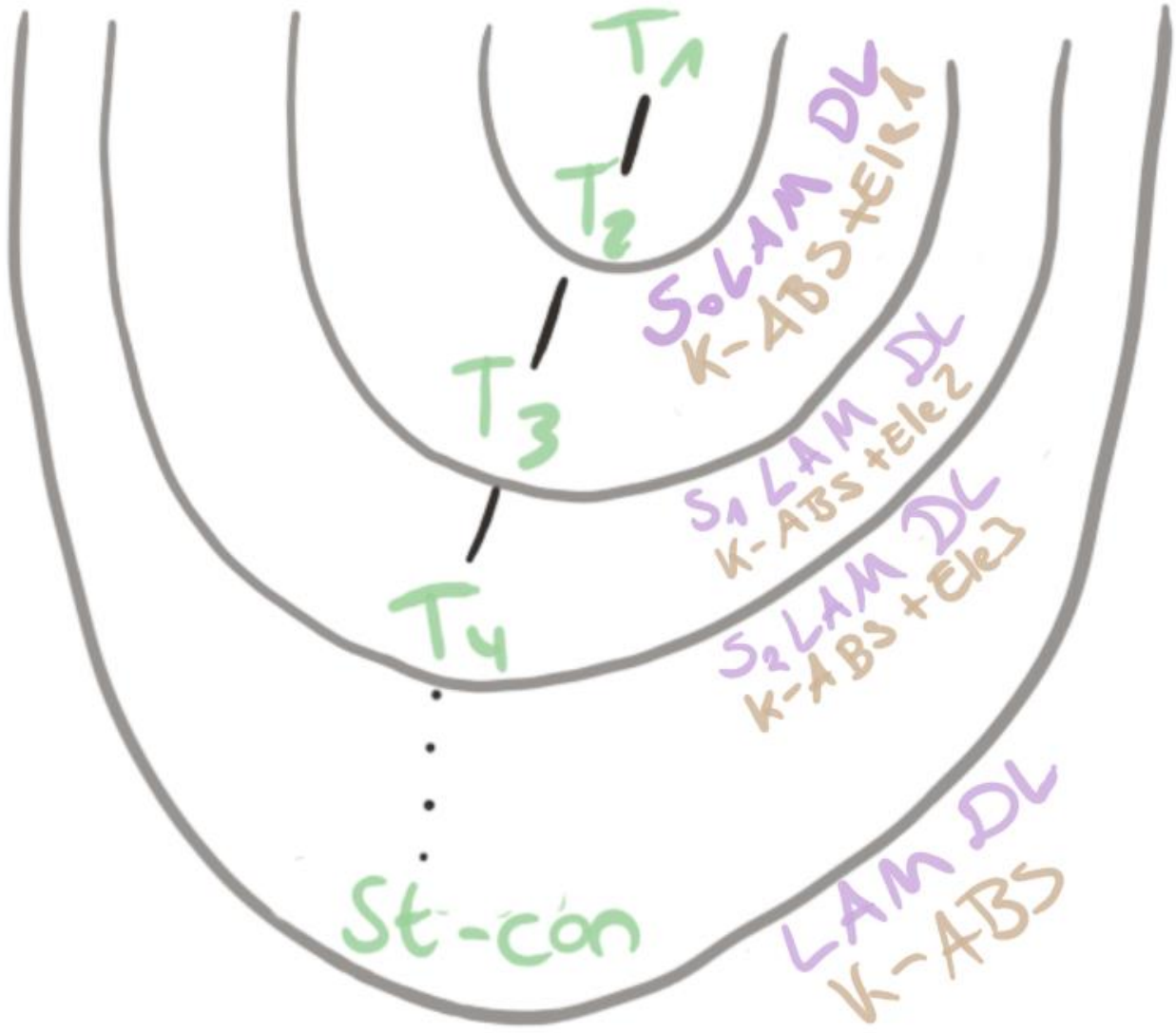
Results



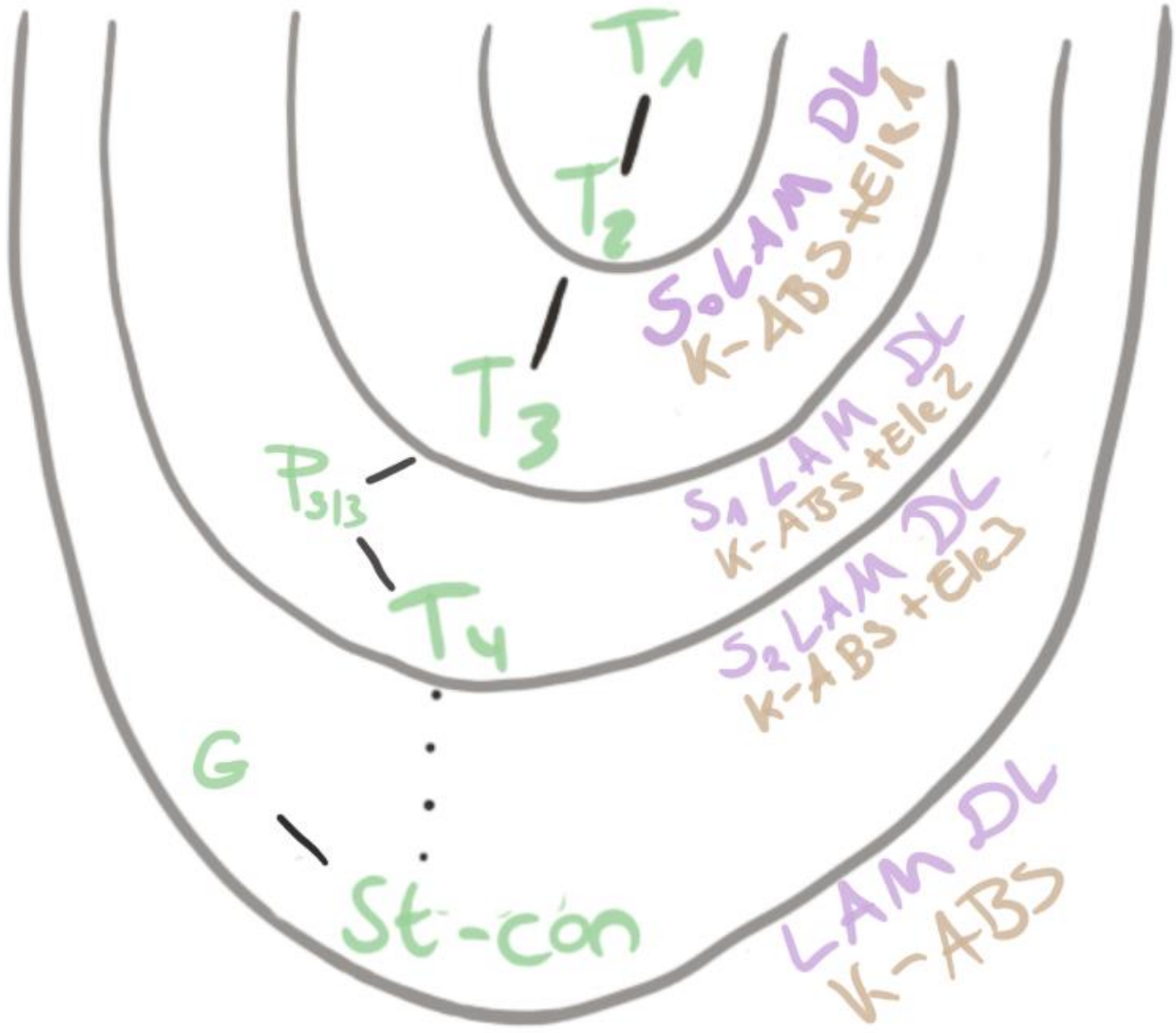
Results



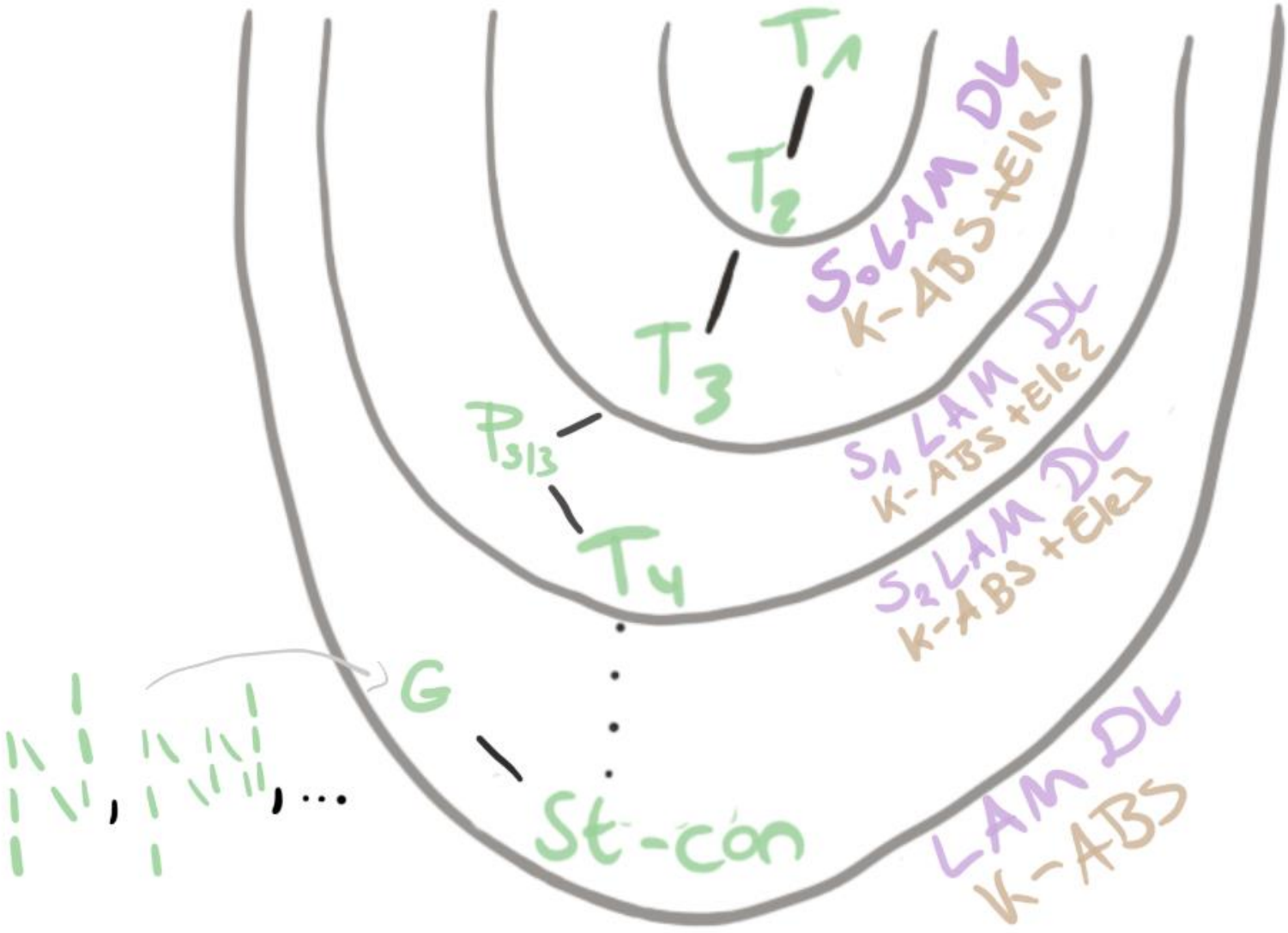
Results


T1
T2
T3
T4
St-con
S0 LAM DL
K-ABS+Ele1
S1 LAM DL
K-ABS+Ele2
S2 LAM DL
K-ABS+Ele3
LAM DL
K-ABS

Results


T1
T2
T3
P313
T4
G
St-con
S0 LAM DL
K-ABS+Ele1
S1 LAM DL
K-ABS+Ele2
S2 LAM DL
K-ABS+Ele3
LAM DL
K-ABS

Results


T1
T2
T3
P313
T4
G
St-con
S0 LAM DL K-ABS+Ele1
S1 LAM DL K-ABS+Ele2
S2 LAM DL K-ABS+Ele3
LAM DL K-ABS

Open Problems

Open Problems

Problem 1: structures above st-con

The following are equivalent

1) CSP(IB) is solved by a $S_{n,m}$ LAM Datalog program
2) Pol(IB) contains K-ABS polymorpisms of all arities and HM(m)
3) IB has n-fixed m-block unfolded caterpillar duality
4) ? $\leq_{pp}$ IB

Open Problems

Problem 2: almost linear Datalog

The following are equivalent

1) CSP(𝔹) is solved by a $\mathrm{SL}^{\mathrm{Fin}}_n\mathrm{AM}$ Datalog program
2) Pol(𝔹) contains ?
3) 𝔹 has unfolded jellyfish duality
4) $\mathbb{B}_{n+3} \leq_{pp} \mathbb{B}$

Open Problems

Problem 2: almost linear Datalog

The following are equivalent

1) CSP(𝔹) is solved by a $\mathrm{SL}^{\mathrm{Fin}}_n$AM Datalog program
2) Pol(𝔹) contains ?
3) 𝔹 has unfolded jellyfish duality



4) $\mathbb{B}_{n+3} \leq_{pp} \mathbb{B}$

Open Problems

Problem 2: almost linear Datalog

The following are equivalent

1) CSP(𝔹) is solved by a $SL^{Fin}_n AM$ Datalog program
2) Pol(𝔹) contains ?
3) 𝔹 has unfolded jellyfish duality
4) $\mathbb{B}_{n+3} \leq_{pp} \mathbb{B}$



Problem 3:

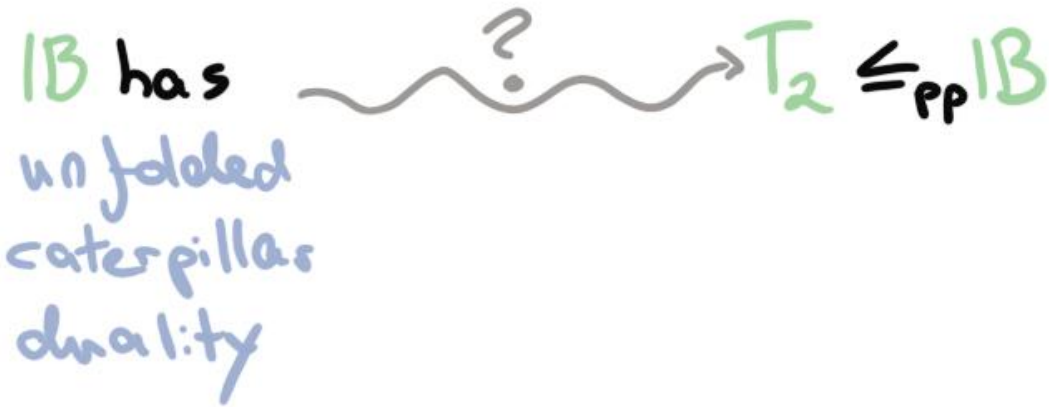
𝔹 is pp-constructed from a structure with finite duality

iff

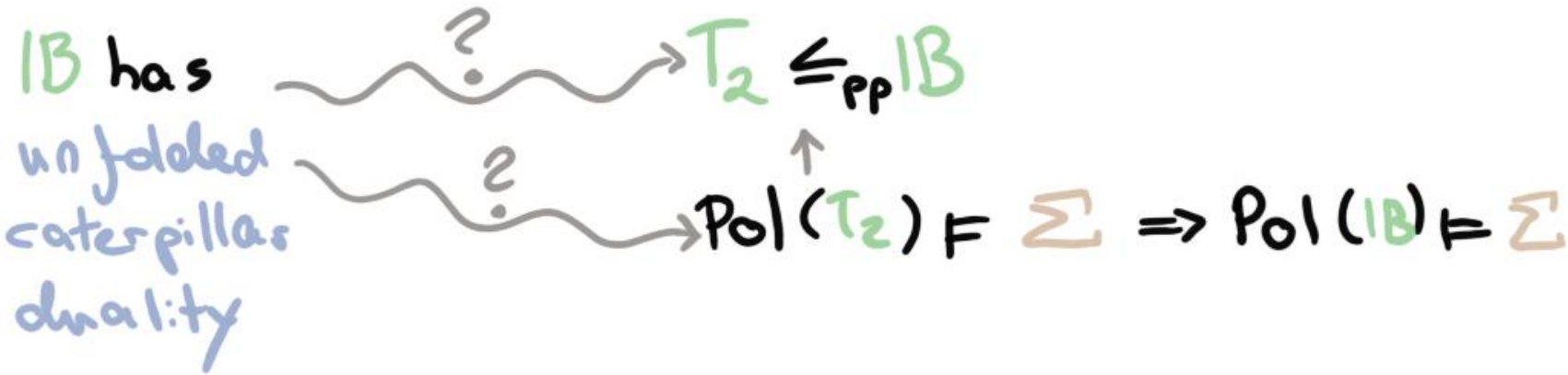
$\exists n,m$ CSP(𝔹) is solved by a $SnLmAM$ Datalog program

From Duality to pp-construction

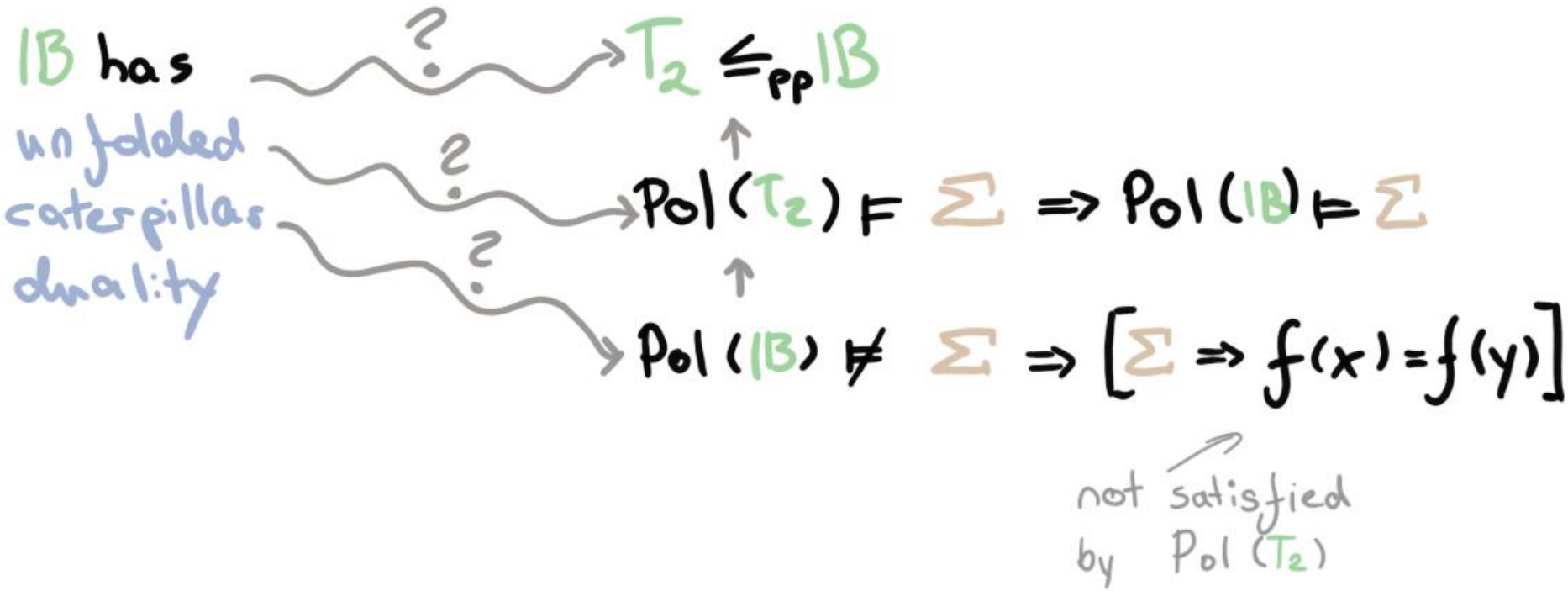
From Duality to pp-construction



From Duality to pp-construction



From Duality to pp-construction



From Duality to pp-construction

From Duality to pp-construction

$\mathrm{Pol}(\mathbb{B}) \not\models \Sigma \Rightarrow \mathrm{Ind}(\mathbb{B}, \Sigma) \not\to \mathbb{B}$

From Duality to pp-construction

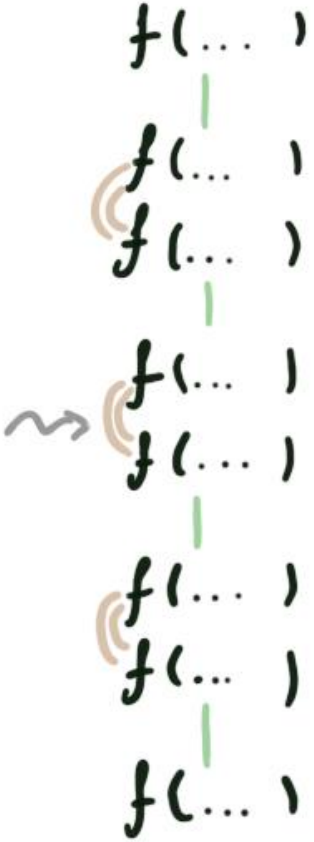
$\mathrm{Pol}(\mathbb{B}) \not\models \Sigma \Rightarrow \mathrm{Ind}(\mathbb{B}, \Sigma) \not\to \mathbb{B}$

$\overset{\text{cpd}}{\Rightarrow}$ 

From Duality to pp-construction

$\mathrm{Pol}(\mathbb{B}) \not\models \Sigma \Rightarrow \mathrm{Ind}(\mathbb{B}, \Sigma) \not\to \mathbb{B}$

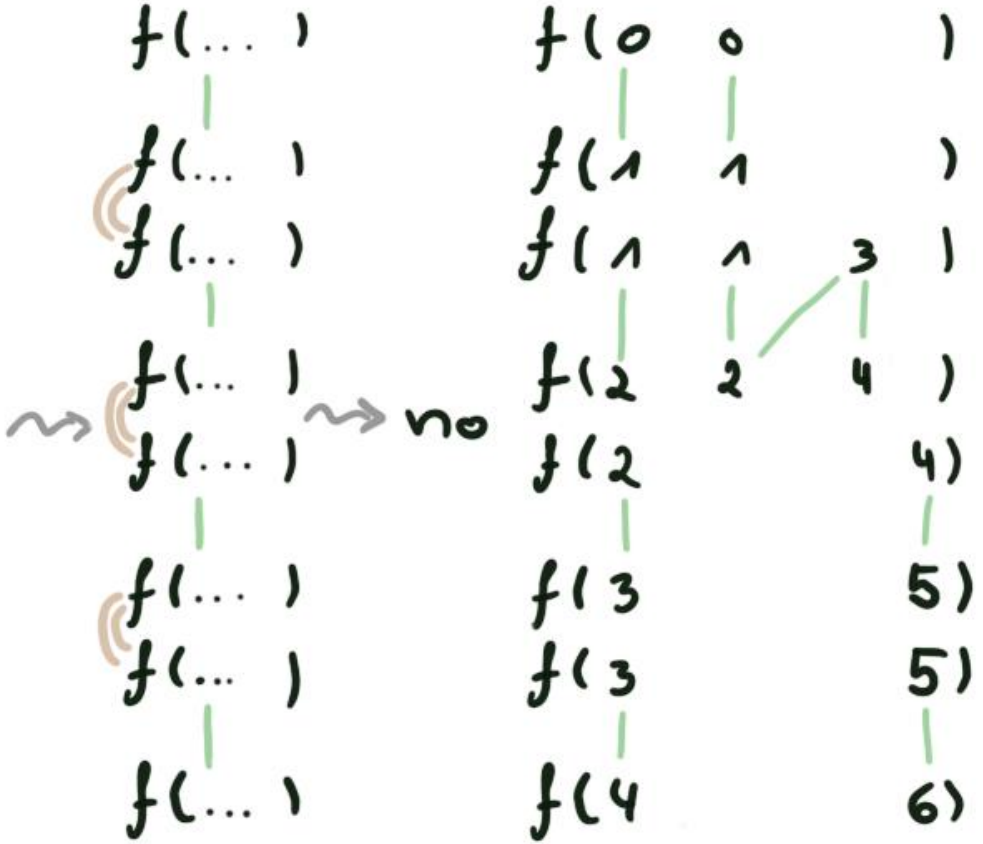
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

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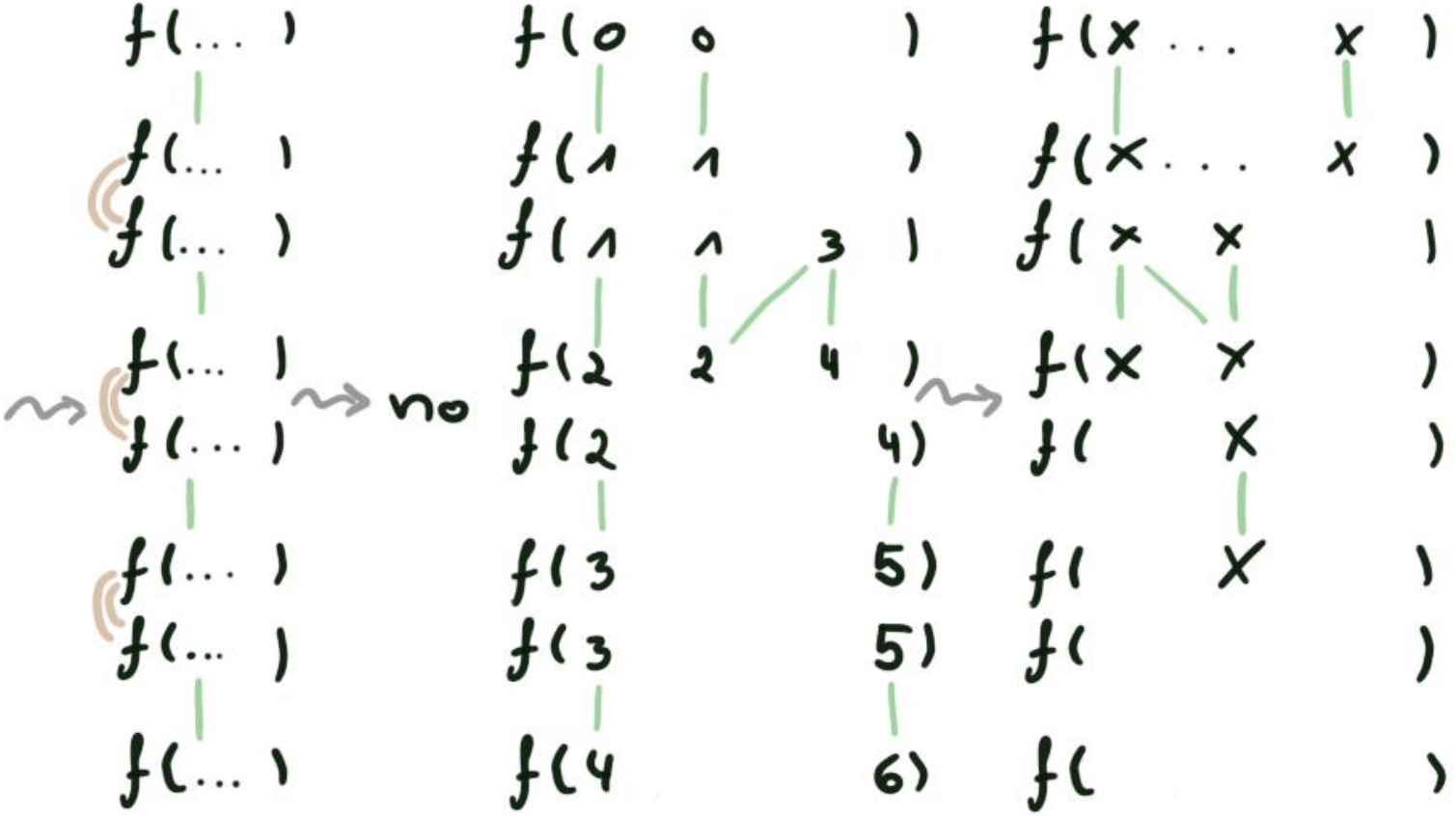
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

$\mathrm{Pol}(\mathbb{B}) \not\models \Sigma \Rightarrow \mathrm{Ind}(\mathbb{B}, \Sigma) \not\to \mathbb{B}$

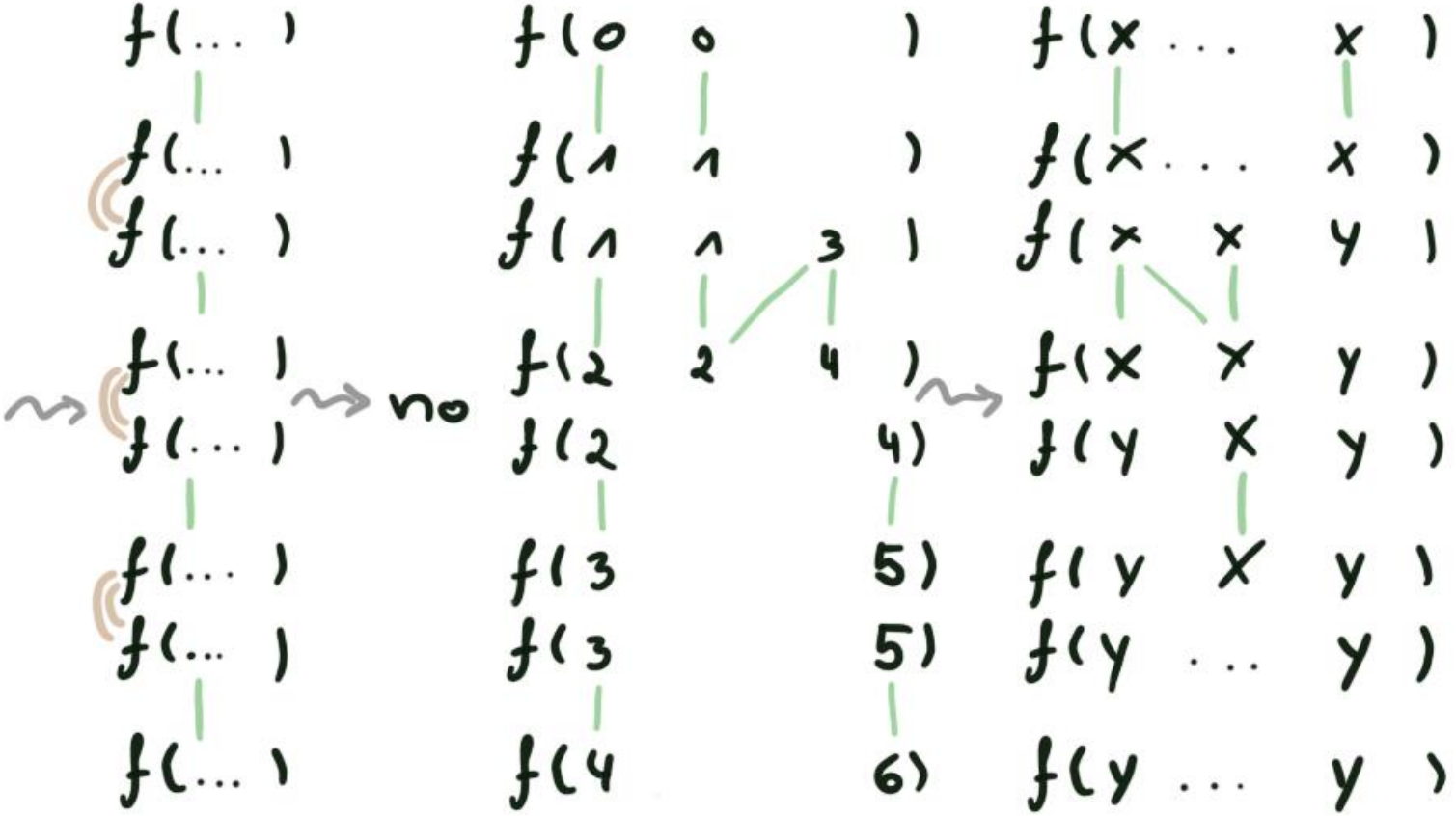
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

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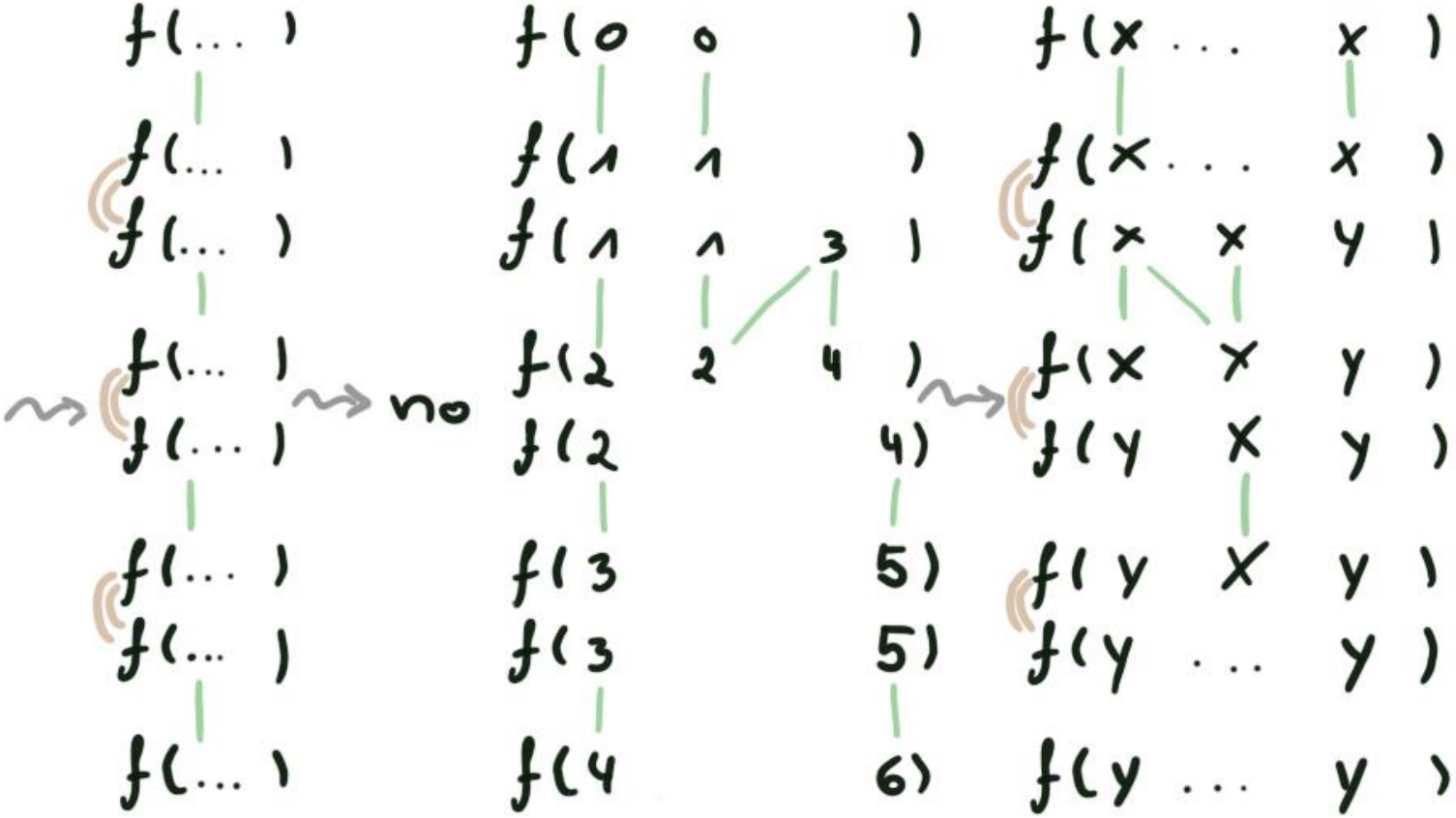
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

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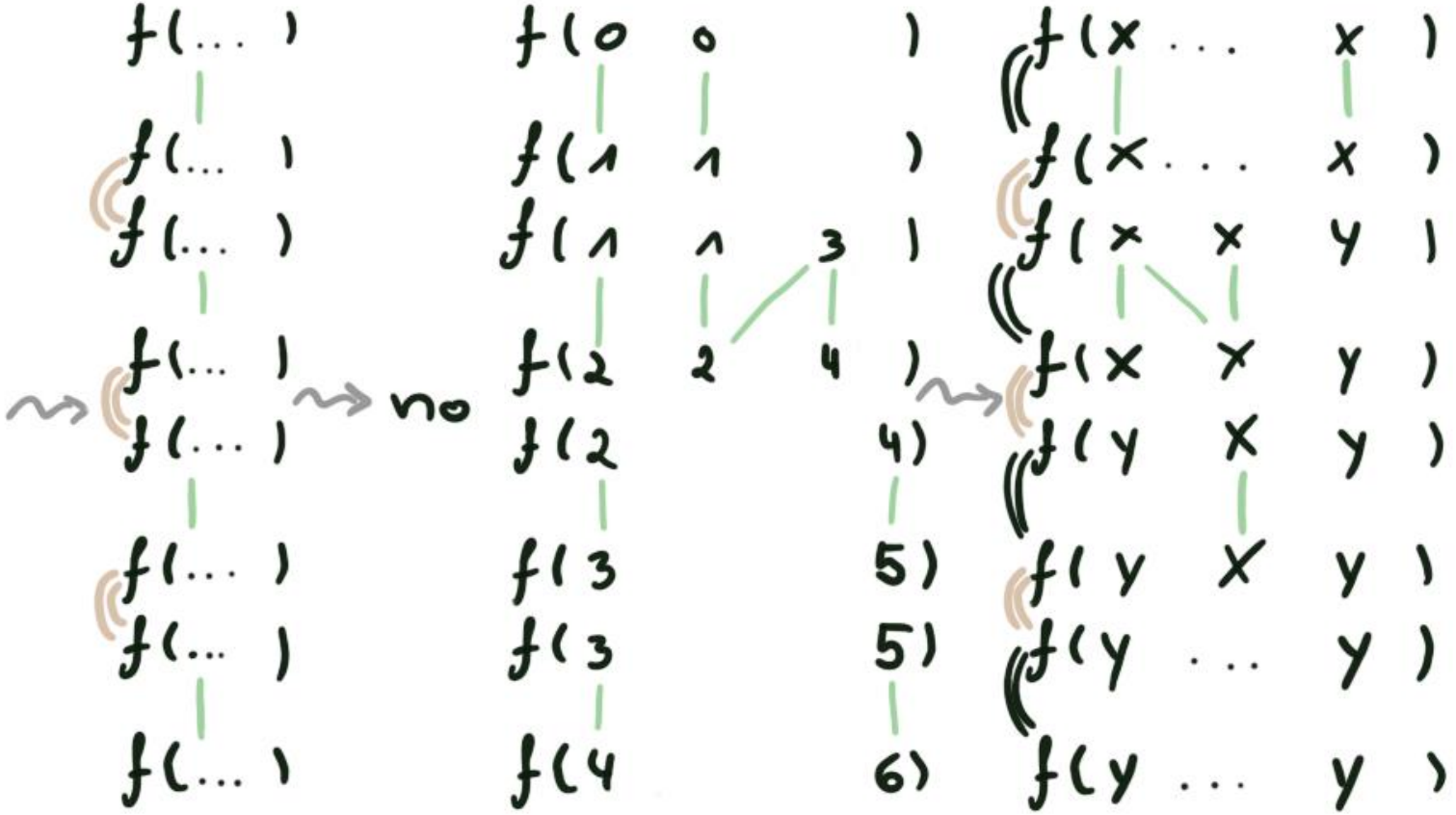
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

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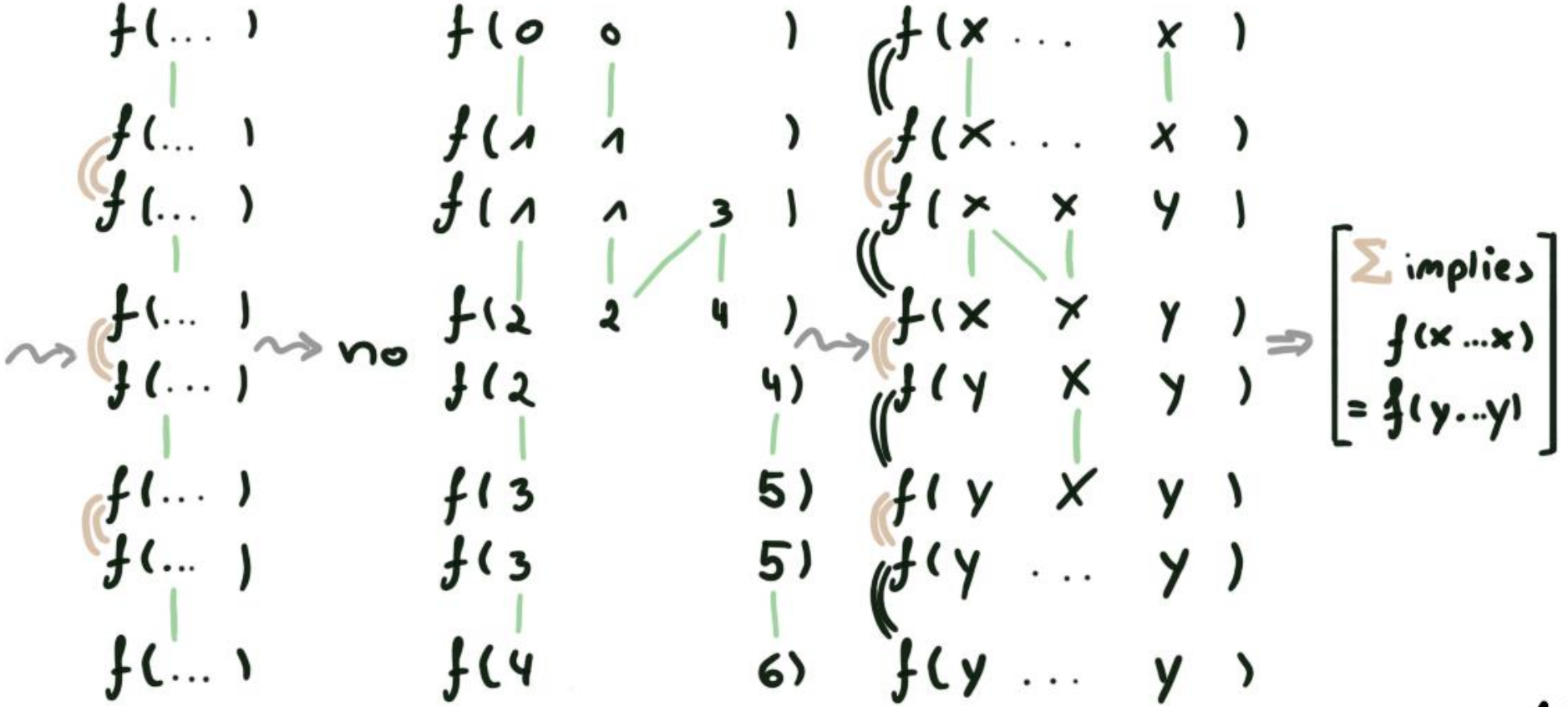
$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction

$\mathrm{Pol}(\mathbb{B}) \not\models \Sigma \Rightarrow \mathrm{Ind}(\mathbb{B}, \Sigma) \not\to \mathbb{B}$

$\overset{\text{cpd}}{\Rightarrow}$ 



From Duality to pp-construction



Paper:

Almost Symmetric Linear Arc Monadic Datalog and Transitive Tournaments

https://arxiv.org/abs/2606.24711

Definition 3.3. For $n \geq 1$, the *elevator condition of length* n, called $\mathrm{Ele}(n)$, consists of the identities

$$e_0(x_1, \ldots, x_n) \approx e_1(x_1, z, z, x_2, \ldots, x_n)$$

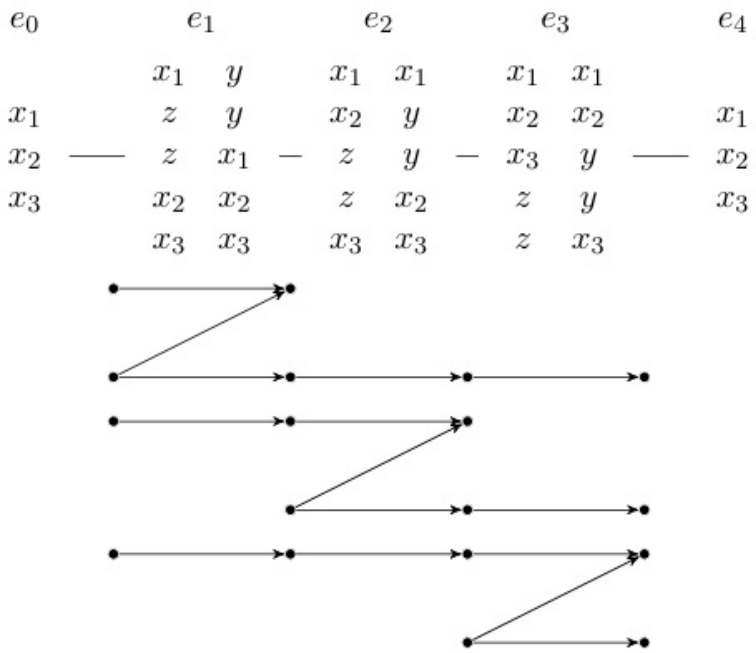
for all $i \in [n-1]$

$$e_i(x_1, \ldots, x_{i-1}, y, y, x_i, x_{i+1}, x_{i+2}, \ldots, x_n) \approx e_{i+1}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, z, z, x_{i+2}, \ldots, x_n)$$

and

$$e_n(x_1, \ldots, x_{n-1}, y, y, x_n) \approx e_{n+1}(x_1, \ldots, x_n).$$

Ele(3)



HM(3)

