On half-homomorphisms of some types of Loops

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Half-homomorphism

- Let $(L, *)$ and $(L', \cdot)$ be two binary algebras. A map $f : L \to L'$ is a half-homomorphism if

$$f(x * y) \in \{f(x) \cdot f(y), f(y) \cdot f(x)\}$$

  for every $x, y \in L$.
- We say that a half-homomorphism is proper if it is neither a homomorphism nor an anti-homomorphism.

Half-homomorphism

- In the year 1957, W.R. Scott proved that there is no proper half-homomorphism between cancellation semi-groups.
- He also gave an example of a loop of order 8 that has a proper half-automorphism, so Scott's result can not be generalized for all loops.

1	2	3	4	5	6	7	8
2	3	8	1	6	5	4	7
3	7	1	8	2	4	5	6
4	6	7	2	1	8	3	5
5	1	2	6	3	7	8	4
6	5	4	7	8	1	2	3
7	8	6	5	4	3	1	2
8	4	5	3	7	2	6	1

Loops

- A *loop* is a nonempty set L with a binary operation $\cdot$ such that, for every a and $b \in L$, the equations

$$a \cdot x = b \text{ and } y \cdot a = b$$

have unique solution and, there exists $1 \in L$ such that

$$1 \cdot x = x \cdot 1 = x, \text{ for all } x \in L.$$

- Groups are, precisely, associative loops.

Loops

- Consider the loop L' given by the following Cayley Table:

1	2	3	4	5	6
2	3	4	5	6	1
3	1	5	6	4	2
4	5	6	1	2	3
5	6	1	2	3	4
6	4	2	3	1	5

- L' is nonassociative $3 \cdot (3 \cdot 3) = 4$ and $(3 \cdot 3) \cdot 3 = 1$.

Properties of half-isomorphism between loops

- Let $f : L \to L'$ be a half-isomorphism and let H be a subloop of L'. Then
  - i) $f^{-1}(H)$ is a subloop of L.
  - ii) If H is commutative, then $f^{-1}(H)$ is commutative and $f^{-1}(H) \cong H$.
  - iii) $f(1_L) = 1_{L'}$.
  - iv) If every element of L' has a two-sided inverse, then every element of L has a two-sided inverse and $x^{-1} = f^{-1}(f(x)^{-1})$.

Properties of half-isomorphism between loops

- Let $f : L \to L'$ be a half-isomorphism and suppose that L' is power associative. Then
  i) $f(x^n) = f(x)^n$, for every $x \in L$ and $n \in \mathbb{Z}$.
  ii) L is power associative.
  iii) If $x \in L$ has finite order, then $o(x) = o(f(x))$

Special half-isomorphism

Definition

Let L, L' be loops. A half-isomorphism $f : L \to L'$ is called special if the inverse mapping $f^{-1} : L' \to L$ is also a half-isomorphism.

Theorem

Let $f : (L, *) \to (L', \cdot)$ be a half-isomorphism. The following are equivalent:

(i) *f is special;*

(ii) *$\{f(x * y), f(y * x)\} = \{f(x) \cdot f(y), f(y) \cdot f(x)\}$, for all $x, y \in L$;*

(iii) *For all $x, y \in L$ with $x * y = y * x$, we have $f(x * y) = f(x) \cdot f(y)$.*

- Let $L = C_6$ be the cyclic group of order 6 and L' be the loop of the previous example.

$L = C_6$

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5

L'

1	2	3	4	5	6
2	3	4	5	6	1
3	1	5	6	4	2
4	5	6	1	2	3
5	6	1	2	3	4
6	4	2	3	1	5

- The map $f : L \to L'$ defined by $f(x) = x$ is a half-isomorphism which is not special.

The half-automorphism group

Proposition

Every half-automorphism of a finite loop is special.

Corollary

For a finite loop L, the set $HAut(L)$ of all half-automorphisms of L is a group.

Loops

- A loop L is said to be
  - a Moufang loop if $(xy)(zx) = (x(yz))x$, for every x, y and $z \in L$;
  - a left Bol loop if $(x(yx))z = x(y(xz))$, for every x, y and $z \in L$;
  - a right Bol loop if $x((yz)y) = ((xy)z)y$, for every x, y and $z \in L$;
  - an automorphic loop if $Inn(L) \leq Aut(L)$.

When does Scott's result hold?

- Moufang loops of odd order - Gagola and Giuliani (2012);
- Finite automorphic Moufang loops - Grichkov, Giuliani, Rasskazova and Sabinina (2016);
- If Q is a Moufang loop such that $Q/N(Q)$ is $2-$divisible, then every half-isomorphism from Q is either an isomorphism or anti-isomorphism - Kinyon, Stuhl and Vojtěchovský (2016);
- Automorphic loops of odd order - Giuliani and dos Anjos (2023).

But...

- Conditions for the existence of proper half-automorphisms for certain Moufang loops of even order, including Chein loops - Gagola and Giuliani (2013);
- Half-automorphism group for a class of automorphic loops of even order - Giuliani and dos Anjos (2020);
- Half-automorphism group for some Chein loops - dos Anjos (to appear)
- Half-automorphism group for some Bol loops - B. and dos Anjos (2022)
- Half-automorphism group of some code loops - B. and Miguel Pires (to appear)

Bol loops

- A *right Bol loop* is a loop that satisfies the right Bol identity

$$x((yz)y) = ((xy)z)y,$$

  and a *left Bol loop* is a loop that satisfies the left Bol identity

$$(x(yx))z = x(y(xz)).$$

- If L is a Bol loop and if it has an anti-automorphism, then L is a Moufang loop.

- If φ is a half-automorphism of a Bol loop L that is not Moufang, then φ is either a proper half-automorphism or an automorphism of L.

Construction of a class of Bol loops

Proposition (Foguel, Kinyon, Phillips)

Let G be a group, $H \leq G$, and $B \subset G$ a right transversal of H in G. If B is a twisted subgroup of G, then B with the operation

$$x \cdot y = z, \text{ if } xy = hz, \text{ for some } h \in H, \tag{1}$$

is a Bol loop. Conversely, if H is core-free and $(B, \cdot)$ is a Bol loop, then B is a twisted subgroup of G.

Construction of a class of Bol loops

- Let M be an abelian group. The *generalized dihedral group of* M can be defined by $D(M) = M \cup Mr$, where $r \notin M$, $r^2 = 1$ and $rxr = x^{-1}$, for every $x \in M$.
- We have that $H = 0 \times 0 \times \{1, r\}$ is a subgroup of order 2 of the direct product $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times D(M)$.
- The set
$$B = \{(0,0,x), (l,s,rx) \,|\, x \in M,\ l,s \in \mathbb{Z}_2, (l,s) \neq (0,0)\}.$$
is a right transversal of H in G which contains $(0,0,1)$, the identity of G. Also, B is a twisted subgroup of G.

Construction of a class of Bol loops

- The set $B = \{(0,0,x), (l,s,rx) \,|\, x \in M,\ l,s \in \mathbb{Z}_2, (l,s) \neq (0,0)\}$ is a Bol loop with the following operation

$$\begin{aligned}
(0,0,x)\cdot(0,0,y) &= (0,0,xy),\\
(0,0,x)\cdot(l,s,ry) &= (l,s,rx^{-1}y),\\
(l,s,rx)\cdot(0,0,y) &= (l,s,rxy),\\
(1,1,rx)\cdot(1,1,ry) &= (0,0,x^{-1}y),\\
(l,s,rx)\cdot(u,v,ry) &= (l+u,s+v,rx^{-1}y).
\end{aligned}$$

Construction of a class of Bol loops

- Let M be an abelian group. Define $L_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times M$ and consider the following operation on L_M:

$$(l, s, x) * (u, v, y) = \begin{cases} (l, s, xy), & \text{if } u = v = 0, \\ (l + u, s + v, x^{-1}y), & \text{otherwise.} \end{cases}$$

The Bol loop

Let M be a finite abelian group with exponent greater than 2 and $L_M = K \times M$, where $K = \{1, a, b, c\}$ is the Klein group. Then the set L_M with the operation

$$\begin{array}{ccccc} (1,x) & * & (1,y) & = & (1,xy) \\ (A,x) & * & (1,y) & = & (A,xy) \\ (1,x) & * & (B,y) & = & (B,x^{-1}y) \\ (A,x) & * & (B,y) & = & (AB,x^{-1}y), \end{array}$$

where $A, B \neq 1$, is a Bol loop which is not a Moufang loop.

Proposition

If M is an elementary abelian 2-group, then $(L_M, *)$ is also an elementary abelian 2-group. If M is not an elementary abelian 2-group, then $(L_M, *)$ is a nonassociative, noncommutative Bol loop, which is not Moufang.

Half-automorphisms of the Bol loop L_M

- Recall that $f : L_M \to L_M$ is a half-automorphism of L_M if f is a bijection of L_M and $f(XY) \in \{f(X)f(Y), f(Y)f(X)\}$, for every X and Y in L_M.

Proposition

Let $(1, M) = \{(1, x) \in L_M; x \in M\}$ and let f be a half-automorphism of L_M. Then $f(1, M) = (1, M)$.

Proposition

Let f be a half-autmorphism of L_M. For every $x \in M$, consider $f^{''}(x) \in M$ as $(1, f^{''}(x)) = f(1, x)$. Then $f'' : M \to M$ is an automorphism of M.

Half-automorphisms of the Bol loop L_M

- Since the set $(K, 1)$ is a subgroup of L_M isomorphic to K, $f(K, 1)$ is a subloop of L_M of order 4, and it is a group isomorphic to K.
- Let $\mathcal{H}_M = \{H \leq L_M \ ; \ H \cong K, |H \cap (1, M)| = 1\}$.

Proposition

Let f be a half-automorphism of L_M. Then $f(K, 1) \in \mathcal{H}_M$.

Half-automorphisms of the Bol loop L_M

Proposition

Let $H \in \mathcal{H}_M$. There are three possibilities:
(i) $H = (K, 1)$,
(ii) $H = \{(1, 1), (A, x), (B, x), (C, 1)\}$, with $\{A, B, C\} = \{a, b, c\}$ and $o(x) = 2$,
(iii) $H = \{(1, 1), (A, x), (B, y), (C, xy)\}$, with $\{A, B, C\} = \{a, b, c\}$ and $x \neq y$ are elements of order equal to 2.
In particular, if M has odd order, then $\mathcal{H}_M = \{(K, 1)\}$.

Half-automorphisms of the Bol loop L_M

Corollary

Let f be a half-automorphism of L_M. Then there exist a unique $f' \in Aut(K)$ and $x, y \in M$ such that $o(x), o(y) \leq 2$ and

$$f(A, 1) = (f'(A), \alpha_{(x,y)}(A)), \text{ for every } A \in K,$$

where $\alpha_{(x,y)}(1) = 1$, $\alpha_{(x,y)}(a) = x$, $\alpha_{(x,y)}(b) = y$ and $\alpha_{(x,y)}(c) = xy$.

Remark

The mapping $\alpha_{(x,y)} : K \to M$ is a homomorphism and $(1, \alpha_{(x,y)}(A)) \in \mathcal{Z}(L_M)$, for every $A \in K$.

Half-automorphisms of the Bol loop L_M

- Every element (A, x) in L_M can be written as $(A, 1)(1, x)$;
- There are $f' \in Aut(K)$ and $u, v \in M$ such that $f(A, 1) = (f'(A), \alpha_{(u,v)}(A))$, where $o(u), o(v) \leq 2$. For $A \neq 1$,

$$f(A, x) = f((A, 1)(1, x)) \in \{(f'(A), 1)(1, f''(x)), (1, f''(x))(f'(A), 1)\},$$

and so

$$f(A, x) \in \{(f'(A), f''(x)\alpha_{(u,v)}(A)), (f'(A), f''(x^{-1})\alpha_{(u,v)}(A))\}.$$

Half-automorphisms of the Bol loop L_M

Then if $f' \in Aut(K)$, $f'' \in Aut(M)$ and $u, v \in M$ are elements such that $o(u), o(v) \leq 2$, we define $F^{+}_{(f',f'',u,v)}, F^{-}_{(f',f'',u,v)} : L_M \to L_M$ by

$$F^{+}_{(f',f'',u,v)}(A,x) = (f'(A), f''(x)\alpha_{(u,v)}(A)) \quad \text{and}$$

$$F^{-}_{(f',f'',u,v)}(A,x) = \begin{cases} (f'(A), f''(x)\alpha_{(u,v)}(A)), & \text{if} \quad A = 1, \\ (f'(A), f''(x^{-1})\alpha_{(u,v)}(A)), & \text{otherwise.} \end{cases}$$

For making the notation easier, we will write $f^{+}_{(u,v)}$ and $f^{-}_{(u,v)}$ instead of $F^{+}_{(f',f'',u,v)}$ and $F^{-}_{(f',f'',u,v)}$, respectively.

Half-automorphisms of the Bol loop L_M

Proposition

$f^+_{(u,v)}$ is an automorphism and $f^-_{(u,v)}$ is a proper half-automorphism of L_M.

Proposition

Let g be an automorphism of L_M. Then $g = g^+_{(u,v)}$.

Proposition

Let g be a proper half-automorphism of L_M. Then $g = g^-_{(u,v)}$.

Half-automorphism group of L_M

Proposition

$Half(L_M) \cong C_2 \times Aut(L_M)$.

Half-automorphism group of L_M

Proposition

Let $\mathcal{A} = \{F^{+}_{(f',f'',1,1)} \,|\, f' \in Aut(K), f'' \in Aut(M)\}$. Then $\mathcal{A} \cong Aut(K) \times Aut(M)$.

- In the case that $|M|$ is odd, then there is no element of order 2 in M.

Theorem

Let $L_M = K \times M$ be the Bol loop where K is the Klein group and M is an abelian group of odd order. Then

$$Aut(L_M) \cong S_3 \times Aut(M) \text{ and } Half(L_M) \cong C_2 \times S_3 \times Aut(M).$$

Half-automorphism group of L_M when $|M|$ is even

- Then $M = C_{2^{i_1}} \times C_{2^{i_2}} \times \ldots \times C_{2^{i_s}} \times M_1$, where M_1 is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all j.
- The set $H = \{x \in M \mid o(x) \leq 2\}$ is a subgroup of M of exponent 2 with $|H| = 2^s$.
- Denote by I_K and I_M the identity mappings of K and M and consider
$$\mathcal{B} = \{F^{+}_{(I_K, I_M, x, y)} \mid x, y \in H\}.$$
subgroup of $Aut(L_M)$.
- Also, $|\mathcal{A} \cap \mathcal{B}| = 1$, and so, $|\mathcal{A}\mathcal{B}| = |\mathcal{A}|.|\mathcal{B}| = 2^{2s}.|Aut(K)|.|Aut(M)|$

Half-automorphism group of L_M

Proposition

Let K be the Klein group, let M be an abelian group of even order and let $L_M = K \times M$ be the associated Bol loop. Then $Aut(L_M) = \mathcal{A}\,\mathcal{B} = \mathcal{B}\,\mathcal{A}$.

Proposition

Using the notation above,

(a) $\mathcal{B} \cong H \times H \cong C_2^{2s}$,

(b) $\mathcal{B} \lhd Aut(L_M)$ and

(c) $Aut(L_M)/\mathcal{B} \cong \mathcal{A}$.

Half-automorphism group of L_M

- $Aut(L_M) \cong \mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B}$.

Theorem

Let M be a finite abelian group of even order and exponent greater than 2. Write $M = C_{2^{i_1}} \times C_{2^{i_2}} \times \ldots \times C_{2^{i_s}} \times M_1$, where M_1 is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all j. Then

$$Aut(L_M) \cong \mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B} \text{ and } Half(L_M) \cong C_2 \times (\mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B}),$$

where $\mathcal{A} \cong S_3 \times Aut(M)$ and $\mathcal{B} \cong C_2^{2s}$.

Example 1

- Let $M = C_3$, the cyclic group of order 3. Then L_M is a nonassociative Bol loop of order 12, which is recognized by the command "RightBolLoop$(12, 3)$" in the library of loops of the LOOPS package.
- Since $Aut(M) = C_2$, we have that
$$Aut(L_M) \cong C_2 \times S_3 \text{ and } Half(L_M) \cong C_2^2 \times S_3$$
- L_M has 24 half-automorphisms, from which 12 are proper.

Example 1

*	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	10	11	9	12	7
3	3	5	6	2	4	1	10	9	12	11	7	8
4	4	6	5	1	3	2	9	11	7	12	8	10
5	5	3	2	6	1	4	12	7	10	8	9	11
6	6	4	1	5	2	3	11	12	8	7	10	9
7	7	9	11	8	12	10	1	5	4	6	3	2
8	8	10	12	7	11	9	2	1	6	5	4	3
9	9	7	8	11	10	12	4	3	1	2	5	6
10	10	8	7	12	9	11	3	2	5	1	6	4
11	11	12	10	9	8	7	6	4	2	3	1	5
12	12	11	9	10	7	8	5	6	3	4	2	1

Example 1

- The automorphisms of L_M are the permutations:
$$I_d,\quad (7,9)(8,11)(10,12),\quad (2,8,11)(4,12,10)(5,9,7),$$
$$(2,9,11,5,8,7)(3,6)(4,12,10),\quad (2,11)(4,10)(5,7),\quad (2,11,8)(4,10,12)(5,7,9),$$
$$(2,5)(3,6)(7,8)(9,11)(10,12),\quad (2,5)(3,6)(7,11)(8,9),\quad (2,7)(3,6)(4,10)(5,11)(8,9),$$
$$(2,7,8,5,11,9)(3,6)(4,10,12),\quad (2,8)(4,12)(5,9),\quad (2,9)(3,6)(4,12)(5,8)(7,11).$$

- The proper half-automorphisms of L_M are the permutations:
$$(2,5)(7,8)(9,11)(10,12),\quad (2,5)(7,11)(8,9),\quad (2,7)(4,10)(5,11)(8,9),$$
$$(2,7,8,5,11,9)(4,10,12),\quad (2,9,11,5,8,7)(4,12,10),\quad (2,9)(4,12)(5,8)(7,11),$$
$$(3,6),\quad (3,6)(7,9)(8,11)(10,12),\quad (2,8,11)(3,6)(4,12,10)(5,9,7),$$
$$(2,8)(3,6)(4,12)(5,9),\quad (2,11)(3,6)(4,10)(5,7),\quad (2,11,8)(3,6)(4,10,12)(5,7,9).$$

Example 2

- Let M be the group $C_4 \times C_2$. This group is recognized by the command “SmallGroup$(8, 2)$” in GAP. The Bol loop L_M is nonassociative and has order 32.
- Since $Aut(M) = D_8$, the dihedral group of order 8, we have that

$$Aut(L_M) \cong (S_3 \times D_8) \overset{\sigma}{\ltimes} C_2^4 \text{ and } Half(L_M) \cong C_2 \times ((S_3 \times D_8) \overset{\sigma}{\ltimes} C_2^4)$$

- L_M has 1536 half-automorphisms, from which 768 are proper.

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