Test

# Unique Polish semigroup topology: novel techniques to crack the semigroup of increasing functions on the rational numbers

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# Outline

1. General situation
2. Unique Polish Property, (Non-)Examples
3. Known Techniques
4. Problem for increasing functions on rational numbers
5. Solution for increasing functions on rational numbers

# Algebraic-topological structures

**General situation**

$(S, \circ, \ldots, \mathcal{T})$: domain $S$ with operations $\circ, \ldots$ and a topology $\mathcal{T}$ such that the operations are continuous w.r.t. the topology

topological groups, topological semigroups, ...

# Algebraic-topological structures

## General situation

$(S, \circ, \ldots, \mathcal{T})$: domain $S$ with operations $\circ, \ldots$ and a topology $\mathcal{T}$ such that the operations are continuous w.r.t. the topology

## Example

$A$ countable

$(A^A, \circ, \mathcal{T}_{pw})$, $\mathcal{T}_{pw}$=pointwise topology, basic open: $\{s : s(\bar{x}) = \bar{y}\}$

finite tuple

... topological semigroup

$(\mathrm{Sym}(A), \circ, .^{-1}, \mathcal{T}_{pw})$, subspace topology

... topological group

In fact: Polish topologies

completely metrizable + separable / second countable

**Question**

Can the topology be reconstructed from compatibility with the operations?

# Reconstruction

→ auto. homeo: $H \leq A^A$, $\mathcal{O} = \mathcal{T}_{pw}$

"unique pointwise-like top."

**Question** End$(\mathbb{Q}, <)$

Can the topology be reconstructed from compatibility with the operations?

**Definition**

$(S, \mathcal{T})$ Polish (semi-)group.
$S$ has the Unique Polish Property (UPP)
$:\Leftrightarrow \mathcal{T}$ is the unique Polish (semi-)group topology on $S$
$\Leftrightarrow \forall (H, \mathcal{O})$ Polish (semi-)group $\forall \varphi\colon S \to H$ isomorphism:
$\varphi\colon (S, \mathcal{T}) \to (H, \mathcal{O})$ homeomorphism

In the following: Aut($\mathbb{A}$), End($\mathbb{A}$) for model-theoretic structures $\mathbb{A}$

strictly increasing bijection — increasing function

# Unique Polish Property

**Examples** 60s 90s 00s

Groups: $\mathrm{Sym}(A)$, Aut(random graph), $\mathrm{Aut}(\mathbb{Q},\leq) = \mathrm{Aut}(\mathbb{Q},<)$

Semigroups: $A^A$, End(random graph) 2020

Gle 10s (even unique second countable Hausdorff)

**Examples**

Groups: $\mathrm{Sym}(A)$, Aut(random graph), $\mathrm{Aut}(\mathbb{Q}, \leq) = \mathrm{Aut}(\mathbb{Q}, <)$

Semigroups: $A^A$, End(random graph)
(even unique second countable Hausdorff)

**Non-Examples**

Groups: $(\mathbb{R}, +)$ $(\mathbb{R}, +) \underset{alg.}{\cong} (\mathbb{R}^2, +)$

Semigroups: $\mathrm{Inj}(A)$, $\mathrm{End}(\mathbb{Q}, <)$, $\mathrm{Surj}(A)$

**Question**

Does $\mathrm{End}(\mathbb{Q}, \leq)$ have UPP?

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Theorem (Pinsker, CS 2023)

$\mathrm{End}(\mathbb{Q}, \leq)$ has UPP.

# Techniques

semigroup

For $A^A$, End(random graph), . . . :

1. $\mathcal{T}_{pw}$ coarser than any Polish semigroup topology $\mathcal{T}_{pw} \subseteq \mathcal{T}$
2. $\mathcal{T}_{pw}$ finer than any Polish semigroup topology $\mathcal{T}_{pw} \supseteq \mathcal{T}$

**Definition**

$S$ ~~semigroup~~ monoid.

Zariski topology on $S$... generated by non-solutions to equations:

$\{s \in S : p_k s p_{k-1} s \ldots s p_0 \neq q_\ell s q_{\ell-1} s \ldots s q_0\}$, $k, \ell \geq 1$, $p_i, q_j \in S$

**Fact**

$\mathcal{T}_{pw} \subseteq \mathcal{T}_{Zariski}$ is coarser than any Hausdorff semigroup topology.

So: $\underbrace{\mathcal{T}_{pw} = \mathcal{T}_{Zariski}}$ $\Rightarrow$ ✓ $\quad \mathcal{T}_{pw} \subseteq \mathcal{T}$

# $\mathcal{T}_{pw}$ coarsest Polish semigroup topology

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**Fact**

$\mathcal{T}_{Zariski}$ is coarser than any Hausdorff semigroup topology.

So: $\mathcal{T}_{pw} = \mathcal{T}_{Zariski} \Rightarrow$ ✓

$\mathcal{T}$

End($\mathbb{Q}$, ≤)

$\mathcal{T}_{pw}$

**Examples**

$A^A$, End(random graph), End($\mathbb{Q}, \leq$), End(compl. $k$-partite graph).
(Pinsker, CS 2023: Sufficient conditions for End($\mathbb{A}$) in terms of $\mathbb{A}$)

**Definition**

$(S, \mathcal{T})$ topological semigroup, $\mathcal{K}$ class of top. semigroups.
$(S, \mathcal{T})$ has automatic continuity (AC) w.r.t. $\mathcal{K}$
$:\Leftrightarrow \forall (H, \mathcal{O}) \in \mathcal{K} \, \forall \varphi \colon S \to H$ homomorphism:
$\varphi \colon (S, \mathcal{T}) \to (H, \mathcal{O})$ continuous

$(S, \mathcal{T}_{pw})$ AC w.r.t. Polish semigroups $\Rightarrow$ ✓

$id : (S, \mathcal{T}_{pw}) \to (S, \mathcal{T})$ $\quad \mathcal{T} \subseteq \mathcal{T}_{pw}$

**Definition**

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$(S, \mathcal{T}_{pw})$ AC w.r.t. Polish semigroups $\Rightarrow$ ✓

Method for $S = \mathrm{End}(\dots)$: Lifting from subset, often $\mathrm{Aut}(\dots)$

eg random graph

Great:

**Theorem (Rosendal, Solecki 2007)**

$(\mathrm{Aut}(\mathbb{Q}, \leq), \mathcal{T}_{pw})$ has AC w.r.t. the class of second countable topological groups.

Great:

**Theorem (Rosendal, Solecki 2007 + Elliott, Jonušas, Mesyan, Mitchell, Morayne, Péresse 2019)**

$(\mathrm{Aut}(\mathbb{Q}, \leq), \mathcal{T}_{pw})$ has AC w.r.t. the class of second countable topological semigroups.

# Situation for $\mathrm{End}(\mathbb{Q}, \leq)$

Great:

**Theorem (Rosendal, Solecki 2007 + Elliott, Jonušas, Mesyan, Mitchell, Morayne, Péresse 2019)**

$(\mathrm{Aut}(\mathbb{Q}, \leq), \mathcal{T}_{pw})$ has AC w.r.t. the class of second countable topological semigroups.

Not great:

**Proposition**

$(\mathrm{End}(\mathbb{Q}, \leq), \mathcal{T}_{pw})$ does not have AC w.r.t. the class of Polish topological semigroups.

# Solution

$\mathcal{T}_{rich} \supseteq \mathcal{T}_{pw}$

Find rich topology $\mathcal{T}_{rich}$ such that:

**Proposition 1**

$(\mathrm{End}(\mathbb{Q}, \leq), \mathcal{T}_{rich})$ has AC w.r.t. the class of second countable topological semigroups.

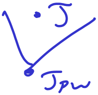
$(\Rightarrow \mathcal{T}_{rich}$ finer than any Polish semigroup topology$)$

**Proposition 2**

$\forall \mathcal{T}$ Polish semigroup topology on $\mathrm{End}(\mathbb{Q}, \leq)$:

$\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{rich} \Rightarrow \mathcal{T} = \mathcal{T}_{pw}$

Zariski:




Prop 1


Prop 2

$\bullet\ \mathcal{T} = \mathcal{T}_{pw}$

# Main instrument

**Definition**

End Aut $\mathcal{T}_{pw}$

$S$ semigroup, $\mathcal{T}$ topology on $S$, $D \subseteq S$, $\mathcal{T}_D$ topology on $D$.
$(S, \mathcal{T})$ has (strong) Property **X** w.r.t. $(D, \mathcal{T}_D)$

$:\Leftrightarrow \exists f, g \in S \, \forall s \in S \, \exists a_s \in D:$ ✓ !

$$s = f a_s g \wedge \forall V \ni a_s \; \mathcal{T}_D\text{-open} \, \exists U \ni s \; \mathcal{T}\text{-open}: \; U \subseteq fVg.$$

# Main instrument

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$:\Leftrightarrow \exists f, g \in S \, \forall s \in S \, \exists a_s \in D :$
$$s = f a_s g \wedge \forall V \ni a_s \ \mathcal{T}_D\text{-open} \, \exists U \ni s \ \mathcal{T}\text{-open} : U \subseteq fVg.$$

**Proposition**
(Elliott, Jonušas, Mesyan, Mitchell, Morayne, Péresse 2019)

$(S, \mathcal{T})$ has Property **X** w.r.t. $(D, \mathcal{T}_D)$ and $(D, \mathcal{T}_D)$ has AC w.r.t. $\mathcal{K}$.
$\Rightarrow (S, \mathcal{T})$ has AC w.r.t. $\mathcal{K}$.

Application: $A^A$, End(random graph), ...

**Definition**

$S$ semigroup, $\mathcal{T}$ topology on $S$, $D \subseteq S$, $\mathcal{T}_D$ topology on $D$.
$(S, \mathcal{T})$ has (strong) Property **X** w.r.t. $(D, \mathcal{T}_D)$

$$:\Leftrightarrow \exists f, g \in S \, \forall s \in S \, \exists a_s \in D:$$
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**Definition**

$S$ monoid, $\mathcal{T}$ topology on $S$, $D \subseteq S$, $\mathcal{T}_D$ topology on $D$.
$(S, \mathcal{T})$ has Pseudo-Property $\overline{\mathbf{X}}$ w.r.t. $(D, \mathcal{T}_D)$
$:\Leftrightarrow \forall s \in S \; \exists e_s, f_s, g_s, h_s \in S \; \exists a_s, b_s \in D:$

$\exists p \in S:$ $pe_s = 1$

$$e_s \text{ left-invertible} \wedge e_s s = f_s a_s h_s b_s g_s \wedge$$
$$\forall V \ni a_s, W \ni b_s \; \mathcal{T}_D\text{-open} \; \exists U \ni s \; \mathcal{T}\text{-open}: \; e_s U \subseteq f_s V h_s W g_s.$$

# Main instrument, continued

**Definition**

$S$ monoid, $\mathcal{T}$ topology on $S$, $D \subseteq S$, $\mathcal{T}_D$ topology on $D$.
$(S, \mathcal{T})$ has Pseudo-Property $\overline{\mathbf{X}}$ w.r.t. $(D, \mathcal{T}_D)$
$:\Leftrightarrow \forall s \in S\, \exists e_s, f_s, g_s, h_s \in S\, \exists a_s, b_s \in D:$
$e_s$ left-invertible $\wedge\ e_s s = f_s a_s h_s b_s g_s\ \wedge$
$\forall V \ni a_s, W \ni b_s$ $\mathcal{T}_D$-open $\exists U \ni s$ $\mathcal{T}$-open: $e_s U \subseteq f_s V h_s W g_s$.

# Main instrument, continued

$\mathcal{T}_{rich}$: $\mathcal{T}_{pw} \cup \Big\{ \{s : s(-\infty,p) \subseteq (-\infty,q]\},\ \{s : s(p,+\infty) \subseteq [q,+\infty)\} : p,q \in \mathbb{Q} \Big\}$ $s(p) \leq q$

$\cup \Big\{ \{s : \sup s = +\infty,\ \inf s \geq -\infty\} \Big\} \cup \Big\{ \{s : \mathrm{Im}(s) \cap (u,v) = \emptyset\} : u,v \in \mathbb{Q} \pm\infty \Big\}$

## Definition

$S$ monoid, $\mathcal{T}$ topology on $S$, $D \subseteq S$, $\mathcal{T}_D$ topology on $D$.
$(S,\mathcal{T})$ has Pseudo-Property $\overline{\mathbf{X}}$ w.r.t. $(D,\mathcal{T}_D)$
$:\Leftrightarrow \forall s \in S\ \exists e_s, f_s, g_s, h_s \in S\ \exists a_s, b_s \in D:$
$e_s$ left-invertible $\wedge\ e_s s = f_s a_s h_s b_s g_s\ \wedge$
$\forall V \ni a_s, W \ni b_s\ \mathcal{T}_D\text{-open}\ \exists U \ni s\ \mathcal{T}\text{-open}:\ e_s U \subseteq f_s V h_s W g_s.$

## Proposition 1

$(\mathrm{End}(\mathbb{Q},\leq), \mathcal{T}_{rich})$ has Pseudo-Property $\overline{\mathbf{X}}$ w.r.t. $(\mathrm{Aut}(\mathbb{Q},\leq), \mathcal{T}_{pw})$.
Hence, $(\mathrm{End}(\mathbb{Q},\leq), \mathcal{T}_{rich})$ has AC w.r.t. the class of second countable topological semigroups.

Zariski: $\mathcal{T}$ $\mathcal{T}_{pw}$

PP $\overline{X}$ $\mathcal{T}_{rich}$ $\mathcal{T}$ $\mathcal{T}_{pw}$

Close the top (Polish)

$\mathcal{T}_{pw} = \mathcal{T}$

# Thank you!