# Abstracting Congruence Extensions through Complete Join–semilattice Morphisms between Commutator Lattices

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# The term condition commutator

Let:
- $\mathcal{V}$: variety of algebras of a similarity type $\tau$
- $A \in \mathcal{V}$

## Definition ( won't be used )

The **term condition commutator** of $A$: $[\cdot, \cdot]_A : \mathrm{Con}(A) \times \mathrm{Con}(A) \to \mathrm{Con}(A)$,

for any $\alpha, \beta \in \mathrm{Con}(A)$,

$$[\alpha, \beta]_A = \overset{(= \min)}{\bigcap} \{\mu \in \mathrm{Con}(A) \mid (\forall\, n, k \in \mathbb{N})\,(\text{for any } (n+k)\text{--ary term } t \text{ over } \tau),$$

$$(\forall\, a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_k, d_1, \ldots, d_k, \in A),$$

$$[((a_1, b_1), \ldots, (a_n, b_n) \in \alpha \text{ and } (c_1, d_1), \ldots, (c_k, d_k) \in \beta) \;\Rightarrow$$

$$((t^A(a_1, \ldots, a_n, c_1, \ldots, c_k), t^A(a_1, \ldots, a_n, d_1, \ldots, d_k)) \in \mu \;\Leftrightarrow$$

$$(t^A(b_1, \ldots, b_n, c_1, \ldots, c_k), t^A(b_1, \ldots, b_n, d_1, \ldots, d_k)) \in \mu)]\}$$

For all $\alpha, \beta, \theta, \zeta \in \mathrm{Con}(A)$: $[\alpha, \beta]_A \subseteq \alpha \cap \beta$ and

$$(\alpha \subseteq \theta \text{ and } \beta \subseteq \zeta) \;\Rightarrow\; [\alpha, \beta]_A \subseteq [\theta, \zeta]_A$$

## The commutator in congruence–modular/distributive varieties

Let:

- $B \in \mathcal{V}$
- $f : A \to B$: morphism in $\mathcal{V}$  (if $A \hookrightarrow B$, then $i_{A,B} : A \to B$ in $\mathcal{V}$)

$$f^* := (f \times f)^{-1} : \mathrm{Con}(B) \to \mathrm{Con}(A).$$

If $\mathcal{V}$ is congruence–modular, then:

- for all $\alpha, \beta \in \mathrm{Con}(A)$ and any $(\beta_i)_{i \in I} \subseteq \mathrm{Con}(A)$: $[\alpha, \beta]_A = [\beta, \alpha]_A$ and

$$[\alpha, \bigvee_{i \in I} \beta_i]_A = \bigvee_{i \in I} [\alpha, \beta_i]_A$$

(equivalent definition for modular commutator below won't be used)

- for each $M \in \mathcal{V}$, $[\cdot, \cdot]_M : \mathrm{Con}(M) \times \mathrm{Con}(M) \to \mathrm{Con}(M)$ is unique s.t., for all $\alpha, \beta \in \mathrm{Con}(M)$, $[\alpha, \beta]_M = \min\{\mu \in \mathrm{Con}(M) \mid \mu \subseteq \alpha \cap \beta, \ (\forall N \in \mathcal{V})\,(\forall h : M \to N)$ ($h$ : surjective morphism in $\mathcal{V} \Rightarrow (\mu \vee \mathrm{Ker}(h) = h^*([h(\alpha \vee \mathrm{Ker}(h)), h(\beta \vee \mathrm{Ker}(h))]_N)))\}$

- so, if $f$ is surjective, then, for all $\alpha, \beta \in \mathrm{Con}(A)$, $Cg_B(f([\alpha, \beta]_A)) =$

$f([\alpha, \beta]_A \vee \mathrm{Ker}(f)) = [f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]_B = [Cg_B(f(\alpha)), Cg_B(f(\beta))]_B$

If $\mathcal{V}$ is congruence–distributive, then $[\cdot, \cdot]_A = \cap$.

# Commutator lattices

## Definition

$(L, \wedge, \vee, [\cdot, \cdot], 0, 1) \stackrel{\text{not.}}{=} (L, [\cdot, \cdot])$ is a **commutator lattice** iff:

- $(L, \wedge, \vee, 0, 1)$ : complete (bounded) lattice

- the **commutator** : $[\cdot, \cdot] : L \times L \to L$ s.t., for all $x, y \in L$ and $(y_i)_{i \in I} \subseteq L$:

$$[x, y] = [y, x] \leq x \wedge y \text{ and } [x, \bigvee_{i \in I} y_i] = \bigvee_{i \in I} [x, y_i]$$

Let:
- $(L, [\cdot, \cdot])$: commutator lattice

Then, for all $a, b, x, y \in L$: $(a \leq x \text{ and } b \leq y) \Rightarrow [a, b] \leq [x, y]$.

$(L, [\cdot, \cdot] = \wedge)$: commutator lattice iff $L$: frame.

- $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_B$ will be commutative and distributive w.r.t. arbitrary $\bigvee$

Thus $(\mathrm{Con}(A), \cap, \vee, [\cdot, \cdot]_A, \Delta_A, \nabla_A)$ and $(\mathrm{Con}(B), \cap, \vee, [\cdot, \cdot]_B, \Delta_B, \nabla_B)$ are

algebraic (i.e. compactly generated) commutator lattices.

# The (minimal) prime spectrum: primes w.r.t. $[\cdot, \cdot]$

The **prime spectrum** of $(L, [\cdot, \cdot])$:

$$\mathrm{Spec}_{(L, [\cdot, \cdot])} := \{p \in L \setminus \{1\} \mid (\forall a, b \in L)([a, b] \leq p \Rightarrow (a \leq p \text{ or } b \leq p))\}$$

$$\mathrm{Spec}(A) := \mathrm{Spec}_{(\mathrm{Con}(A), [\cdot, \cdot]_A)}$$

The **minimal prime spectrum** of $(L, [\cdot, \cdot])$:

$$\mathrm{Min}_{(L, [\cdot, \cdot])} := Min(\mathrm{Spec}_{(L, [\cdot, \cdot])})$$

$$\mathrm{Min}(A) := \mathrm{Min}_{(\mathrm{Con}(A), [\cdot, \cdot]_A)}$$

$$\mathrm{Max}_L := Max(L \setminus \{1\}) \quad (\neq \emptyset \text{ if } |L| > 1 \text{ and } 1 \in \mathrm{Cp}(L) ;$$
$$\subseteq \mathrm{Spec}_{(L, [\cdot, \cdot])} \text{ if } [1, 1] = 1 )$$

$$\mathrm{Max}(A) := \mathrm{Max}_{\mathrm{Con}(A)}$$

- If $\mathcal{V}$: semidegenerate, i.e.: $S \hookrightarrow M$ in $\mathcal{V}$ and $|M| > 1 \Rightarrow |S| > 1$, then $\nabla_A \in \mathrm{Cp}(\mathrm{Con}(A))$.

- If $\mathcal{V}$: congruence–modular and semidegenerate, then $[\alpha, \nabla_A]_A = \alpha$ for all $\alpha \in \mathrm{Con}(A)$, in particular $[\nabla_A, \nabla_A]_A = \nabla_A$.

# Stone/spectral topology on (minimal) primes

The **Stone/spectral topology** on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$:
$$\mathcal{S}_{\mathrm{Spec},(L,[\cdot,\cdot])} = \{D_{(L,[\cdot,\cdot])}(x) \mid x \in L\}$$

with closed sets: $\{V_{(L,[\cdot,\cdot])}(x) \mid x \in L\}$, where, for each $x \in L$:

- $D_{(L,[\cdot,\cdot])}(x) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus [x)_L = \{p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \mid x \nleq p\}$
- $V_{(L,[\cdot,\cdot])}(x) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \cap [x)_L = \{p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \mid x \leq p\}$

having the following basis if $L$ is algebraic: $\{D_{(L,[\cdot,\cdot])}(x) \mid x \in \mathrm{Cp}(L)\}$
$$\mathcal{S}pec_{(L,[\cdot,\cdot])} := (\mathrm{Spec}_{(L,[\cdot,\cdot])}, \mathcal{S}_{\mathrm{Spec},(L,[\cdot,\cdot])})$$

The **Stone/spectral topology** on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ (induced by the one above;

similarly for $\mathrm{Max}_L$):
$$\mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])} = \{D_{(L,[\cdot,\cdot])}(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in L\}$$

with closed sets: $\{V_{(L,[\cdot,\cdot])}(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in L\}$

having this basis if $L$ is algebraic: $\{D_{(L,[\cdot,\cdot])}(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in \mathrm{Cp}(L)\}$
$$\mathcal{M}in_{(L,[\cdot,\cdot])} := (\mathrm{Min}_{(L,[\cdot,\cdot])}, \mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])})$$

The **flat/inverse topology** on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ :

$$\mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])}, \text{ generated by: } \{V_{(L,[\cdot,\cdot])}(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in \mathrm{Cp}(L)\} ,$$

which is a basis for $\mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])}$ .

$$\mathcal{M}in_{(L,[\cdot,\cdot])}^{-1} := (\mathrm{Min}_{(L,[\cdot,\cdot])}, \mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])})$$

If $L = \mathrm{Cp}(L)$ , then the **flat/inverse topology** on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ :

$$\mathcal{F}_{\mathrm{Spec},(L,[\cdot,\cdot])} = \{V_{(L,[\cdot,\cdot])}(x) \mid x \in L\} \; ; \; \mathcal{S}pec_{(L,[\cdot,\cdot])}^{-1} := (\mathrm{Spec}_{(L,[\cdot,\cdot])}, \mathcal{F}_{\mathrm{Spec},(L,[\cdot,\cdot])}) .$$

- $\mathcal{M}in(A) := (\mathrm{Min}(A), \mathcal{S}_{\mathrm{Min},(\mathrm{Con}(A),[\cdot,\cdot]_A)})$

- $\mathcal{M}in^{-1}(A) := (\mathrm{Min}(A), \mathcal{F}_{\mathrm{Min},(\mathrm{Con}(A),[\cdot,\cdot]_A)})$

- $\mathcal{S}pec(A) := (\mathrm{Spec}(A), \mathcal{S}_{\mathrm{Spec},(\mathrm{Con}(A),[\cdot,\cdot]_A)})$

- If $\mathrm{Con}(A)$ is compact , then $\mathcal{S}pec^{-1}(A) := (\mathrm{Spec}(A), \mathcal{F}_{\mathrm{Spec},(\mathrm{Con}(A),[\cdot,\cdot]_A)})$ .

# Implication($\rightarrow$), polar($^\perp$), a commutator lattice congruence

For any $a, b \in L$:  $a \rightarrow b := \bigvee \{x \in L \mid [a, x] \leq b\} = \max\{x \in L \mid [a, x] \leq b\}$

$a^\perp := a \rightarrow 0 = \max\{x \in L \mid [a, x] = 0\}$

$\equiv_{(L, [\cdot, \cdot])} := \{(x, y) \in L \times L \mid V_{(L, [\cdot, \cdot])}(x) = V_{(L, [\cdot, \cdot])}(y)\} \in \mathrm{Con}(L, [\cdot, \cdot])$,

preserves arbitrary $\bigvee$

and $(\forall x, y \in L) ([x, y] \equiv_{(L, [\cdot, \cdot])} x \wedge y)$

so:

- $L/\equiv_{(L, [\cdot, \cdot])}$ is a frame

- if $\mathrm{Cp}(L)$ is closed w.r.t. $[\cdot, \cdot]$, then (since it's closed w.r.t. $\vee$):
  $\mathrm{Cp}(L)/\equiv_{(L, [\cdot, \cdot])}$ is a sublattice of $L/\equiv_{(L, [\cdot, \cdot])}$

If $\mathrm{Cp}(\mathrm{Con}(A))$ is closed w.r.t. $[\cdot, \cdot]_A$, $\nabla_A \in \mathrm{Cp}(\mathrm{Con}(A))$ and
$[\nabla_A, \nabla_A]_A = \nabla_A$, then:

$\mathrm{Cp}(\mathrm{Con}(A))/\equiv_{(\mathrm{Con}(A), [\cdot, \cdot]_A)}$ is the reticulation of $A$,

that is $\mathcal{S}pec(A)$ is homeomorphic to $\mathcal{S}pec_{(\mathrm{Id}(\mathrm{Cp}(\mathrm{Con}(A))/\equiv_{(\mathrm{Con}(A), [\cdot, \cdot]_A)}), \cap)}$.

$(1 \in \mathrm{Cp}(L)$ and $[1, 1] = 1) \Rightarrow 1/\equiv_{(L, [\cdot, \cdot])} = \{1\} \Leftrightarrow (\forall x \in L \setminus \{1\}) (V_{(L, [\cdot, \cdot])}(x) \neq \emptyset)$

## Some conditions

### Conditions

Condition (CP) :

- $L$ is algebraic

- $\mathrm{Cp}(L)$ is closed w.r.t. $[\cdot, \cdot]$

- $1/\equiv_{(L,[\cdot,\cdot])} = \{1\}$

Condition (ID) :

$$(\forall\, p \in \mathrm{Min}_{(L/\equiv_{(L,[\cdot,\cdot])}, \wedge)})\ \left((p]_{L/\equiv_{(L,[\cdot,\cdot])}} \in \mathrm{Min}_{(\mathrm{Id}(L/\equiv_{(L,[\cdot,\cdot])}), \cap)}\right)$$

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$(\ L = \mathrm{Cp}(L)$ and $[1,1] = 1\ ) \Rightarrow (\ L = \mathrm{Cp}(L)$ and $1/\equiv_{(L,[\cdot,\cdot])} = \{1\}\ ) \Rightarrow (L, [\cdot, \cdot]) \vDash (\mathrm{CP})$

$$L = \mathrm{Cp}(L) \;\Rightarrow\; (L, [\cdot, \cdot]) \vDash (\mathrm{ID})$$

If $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ , then: $\begin{cases} (\forall\, a, b \in L)\,([a, b] = 0 \Leftrightarrow a \wedge b = 0) \\ 0/\equiv_{(L,[\cdot,\cdot])} = \{0\} \end{cases}$

$A$ is said to be **semiprime** iff $\bigcap \mathrm{Spec}_{(\mathrm{Con}(A), [\cdot,\cdot]_A)} = \Delta_A$ .

Let:

- $(M, \langle \cdot, \cdot \rangle)$ : commutator lattice
- $h : L \to M$ s.t. $h$ preserves arbitrary $\bigvee$ (so $h$ $\nearrow$ )

Then $h$ has a unique right adjoint $h_*$ , i.e. map $h_* : M \to L$ satisfying:

$$(\forall a \in L)(\forall b \in M)(h(a) \leq b \Leftrightarrow a \leq h_*(b)) ,$$

namely: for all $b \in M$, $h_*(b) = \bigvee \{a \in L \mid h(a) \leq b\} = \max\{a \in L \mid h(a) \leq b\}$ .

Recall that $f : A \to B$: morphism in $\mathcal{V}$ .

$f^\bullet : \mathrm{Con}(A) \to \mathrm{Con}(B)$ , for all $\alpha \in \mathrm{Con}(A)$, $f^\bullet(\alpha) = Cg_B(f(\alpha))$

Then:

- $f^\bullet$ preserves arbitrary $\bigvee$
- the right adjoint of $f^\bullet$ is $f^*$ :
  $$(f^\bullet)_* = f^* = (f \times f)^{-1} : \mathrm{Con}(B) \to \mathrm{Con}(A)$$
- $(\forall a, b \in A)(f^\bullet(Cg_A(a, b)) = Cg_B(f(a), f(b)))$
- hence $f^\bullet(\mathrm{Cp}(\mathrm{Con}(A))) \subseteq \mathrm{Cp}(\mathrm{Con}(B))$

### Conditions

For any $S \subseteq L$ :

- condition $[,]_{\leq}(S)$ : $(\forall \, a, b \in S) \, (h([a, b]) \leq \langle h(a), h(b) \rangle)$

- condition $[,]=(S)$ : $(\forall \, a, b \in S) \, (h([a, b]) = \langle h(a), h(b) \rangle)$

If $S \subseteq T \subseteq L$ s.t. $S$ is $\vee$–dense in $T$ , then:

$$h \vDash [,]_{\leq}(S) \Leftrightarrow h \vDash [,]_{\leq}(T) \text{ and } h \vDash [,]=(S) \Leftrightarrow h \vDash [,]=(T) .$$

In particular, if $L$ is algebraic , then:

$$h \vDash [,]_{\leq}(\mathrm{Cp}(L)) \Leftrightarrow h \vDash [,]_{\leq}(L) \text{ and } h \vDash [,]=(\mathrm{Cp}(L)) \Leftrightarrow h \vDash [,]=(L) .$$

If $\mathcal{V}$ is congruence–modular , then $f^{\bullet} \vDash [,]_{\leq}(\mathrm{Con}(A))$ .

If $\mathcal{V}$ is congruence–modular and $f$ is surjective , then $f^{\bullet} \vDash [,]=(\mathrm{Con}(A))$ .

## Proposition

Let $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$. Assume that $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ and let:

   ① $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$ ;

   ② $(\forall\, a \in \mathrm{Cp}(L))\ (a \leq p \Rightarrow a^{\perp} \not\leq p)$ ;     ④ $(\forall\, a \in L)\ (a \leq p \Rightarrow a^{\perp} \not\leq p)$ ;

   ③ $(\forall\, a \in \mathrm{Cp}(L))\ (a \leq p \Leftrightarrow a^{\perp} \not\leq p)$ ;     ⑤ $(\forall\, a \in L)\ (a \leq p \Leftrightarrow a^{\perp} \not\leq p)$ .

- If $(L,[\cdot,\cdot]) \vDash (\mathrm{CP})$ , then ①$\Leftrightarrow$②$\Leftrightarrow$③ .

- If $(L,[\cdot,\cdot]) \vDash (\mathrm{ID})$ , then ①$\Leftrightarrow$④$\Leftrightarrow$⑤ .

- Thus, if $(L,[\cdot,\cdot]) \vDash (\mathrm{CP}) \& (\mathrm{ID})$ , in particular if $L = \mathrm{Cp}(L)$ and $[1,1] = 1$ , then ①$\Leftrightarrow$②$\Leftrightarrow$③$\Leftrightarrow$④$\Leftrightarrow$⑤ .

# Lattice structures of the Stone/flat topologies on (minimal) primes as bounded sublattices of the power sets of the (minimal) prime spectra

## Proposition

- $L/\equiv_{(L,[\cdot,\cdot])} \cong \mathcal{S}_{\mathrm{Spec},(L,[\cdot,\cdot])} \hookrightarrow \mathcal{P}(\mathrm{Spec}_{(L,[\cdot,\cdot])})$ , hence:

- $\mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])} \hookrightarrow \mathcal{P}(\mathrm{Min}_{(L,[\cdot,\cdot])})$ :
  bounded sublattices closed w.r.t. arbitrary $\bigcup$.

Thus $\mathcal{S}_{\mathrm{Spec},(L,[\cdot,\cdot])}$ and $\mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])}$ are frames .

## Proposition

- $|L| < \infty \ \Rightarrow \ |\mathrm{Min}_{(L,[\cdot,\cdot])}| < \infty \ \Rightarrow \ \mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])} = \mathcal{P}(\mathrm{Min}_{(L,[\cdot,\cdot])})$ .

- $|L| < \infty \ \Rightarrow (\ |\mathrm{Min}_{(L,[\cdot,\cdot])}| < \infty \ $ and $\ L = \mathrm{Cp}(L)\ ) \Rightarrow$

  $(\ |\mathrm{Min}_{(L,[\cdot,\cdot])}| < \infty \ $ and $\ (\forall\, S \subseteq \mathrm{Min}_{(L,[\cdot,\cdot])})(\bigwedge S \in \mathrm{Cp}(L))\ ) \Rightarrow$

  $\mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])} = \mathcal{P}(\mathrm{Min}_{(L,[\cdot,\cdot])})$ .

## Proposition (flat versus Stone topologies on minimal primes)

If $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ and $(L,[\cdot,\cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$, then:

1. $\mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])} \subseteq \mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])}$;

2. if $\mathrm{Cp}(L)$ is closed w.r.t. $\cdot^{\perp}$, in particular if $L = \mathrm{Cp}(L)$, then
   $\mathcal{F}_{\mathrm{Min},(L,[\cdot,\cdot])} = \mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])}$, that is:

   $$\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])} = \mathcal{M}in_{(L,[\cdot,\cdot])}.$$

## Lemma (abstract version of the reticulation property)

If $(L,[\cdot,\cdot]) \vDash (\mathrm{CP})$, then:

1. $\mathcal{S}pec_{(L,[\cdot,\cdot])}$ and $\mathcal{S}pec_{(\mathrm{Id}(\mathrm{Cp}(L)/\equiv_{(L,[\cdot,\cdot])}),\cap)}$ are homeomorphic;

2. $\mathcal{M}in_{(L,[\cdot,\cdot])}$ and $\mathcal{M}in_{(\mathrm{Id}(\mathrm{Cp}(L)/\equiv_{(L,[\cdot,\cdot])}),\cap)}$ are homeomorphic;

3. $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ and $\mathcal{M}in^{-1}_{(\mathrm{Id}(\mathrm{Cp}(L)/\equiv_{(L,[\cdot,\cdot])}),\cap)}$ are homeomorphic.

# Compactness and separability of the Stone/flat topologies

## Proposition

If $(L, [\cdot, \cdot]) \vDash (\mathrm{CP})$, then:

$$\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])} \text{ is compact } \mathrm{T}_1.$$

## Theorem

If $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ and $(L, [\cdot, \cdot]) \vDash (\mathrm{CP})$, then the following are equivalent:

1. $\mathcal{M}in_{(L,[\cdot,\cdot])} = \mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ ;

2. $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is compact ;

3. $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is compact $\mathrm{T}_1$ ;

4. $(\forall\, a \in \mathrm{Cp}(L))\,(\exists\, b \in \mathrm{Cp}(L)), (b \leq a^{\perp} \text{ and } (a \vee b)^{\perp} = 0)$ .

$$\Uparrow$$
$$\mathrm{Cp}(L) \text{ is closed w.r.t. } \cdot^{\perp}$$

### Proposition

If $1 \in \mathrm{Cp}(L)$ and $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is an antichain, then

$\mathcal{M}in_{(L,[\cdot,\cdot])}(= \mathcal{S}pec_{(L,[\cdot,\cdot])})$ is compact.

### Theorem

If:

- $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ ,

- $\mathrm{Cp}(L)$ is closed w.r.t. $\cdot^{\perp}$ and

- $(L, [\cdot, \cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$ ,

in particular if $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ and $L = \mathrm{Cp}(L)$ ,

then $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is Hausdorff and consists solely of clopens, thus

$\mathcal{S}_{\mathrm{Min},(L,[\cdot,\cdot])}$ is a complete Boolean sublattice of $\mathcal{P}(\mathrm{Min}_{(L,[\cdot,\cdot])})$ .

## Lemma ("surjectivity" of $h_*$ on minimal primes)

If $L$ and $M$ are algebraic, $1 \in \mathrm{Cp}(M)$, $h(\mathrm{Cp}(L)) \subseteq \mathrm{Cp}(M)$,
$h \vDash [,]_{\leq}(\mathrm{Cp}(L))$, then:

- if $h_*(\mathrm{Spec}_{(M,\langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$, then $h_*(\mathrm{Min}_{(M,\langle \cdot, \cdot \rangle)}) \supseteq \mathrm{Min}_{(L,[\cdot,\cdot])}$
- if $h_*(\mathrm{Min}_{(M,\langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Min}_{(L,[\cdot,\cdot])}$, then $h_*(\mathrm{Min}_{(M,\langle \cdot, \cdot \rangle)}) = \mathrm{Min}_{(L,[\cdot,\cdot])}$

We call $f : A \to B$ :
- **admissible** iff $f^*(\mathrm{Spec}_{(\mathrm{Con}(B),[\cdot,\cdot]_B)}) \subseteq \mathrm{Spec}_{(\mathrm{Con}(A),[\cdot,\cdot]_A)}$
- $\mathrm{Min}$**–admissible** iff $f^*(\mathrm{Min}_{(\mathrm{Con}(B),[\cdot,\cdot]_B)}) \subseteq \mathrm{Min}_{(\mathrm{Con}(A),[\cdot,\cdot]_A)}$

If $A \hookrightarrow B$, then:
- $A \hookrightarrow B$ is said to be **admissible** iff $i_{A,B}$ is admissible
- $A \hookrightarrow B$ is said to be $\mathrm{Min}$**–admissible** / an $m$**–extension** iff $i_{A,B}$ is $\mathrm{Min}$–admissible
- $i_{A,B}^*(\beta) = \beta \cap \nabla_A$ for all $\beta \in \mathrm{Con}(B)$, thus, by the previous lemma, if $\nabla_B \in \mathrm{Cp}(\mathrm{Con}(B))$ and $f^\bullet \vDash [,]_{\leq}(\mathrm{Cp}(\mathrm{Con}(A)))$, in particular if $\mathcal{V}$ is congruence–modular and semidegenerate, then:

  if $A \hookrightarrow B$ is admissible or $\mathrm{Min}$–admissible, then $A \hookrightarrow B$ satisfies property Lying Over for minimal prime congruences

*If $L$ and $M$ are algebraic , $1 \in \mathrm{Cp}(M)$ , $h(\mathrm{Cp}(L)) \subseteq \mathrm{Cp}(M)$ , $h \vDash [,]_{\leq}(\mathrm{Cp}(L))$*

*and $h_*(\mathrm{Min}_{(M,\langle\cdot,\cdot\rangle)}) \subseteq \mathrm{Min}_{(L,[\cdot,\cdot])}$ , then:*

1. $h_* : \mathcal{M}in_{(M,\langle\cdot,\cdot\rangle)} \to \mathcal{M}in_{(L,[\cdot,\cdot])}$ *is continuous ;*

2. $h_* : \mathcal{M}in^{-1}_{(M,\langle\cdot,\cdot\rangle)} \to \mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ *is continuous ;*

3. *if one of the following holds:*
   - $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ *and* $(L,[\cdot,\cdot]) \vDash ((\mathrm{CP})$ *or* $(\mathrm{ID}))$ ,
   - $\bigwedge \mathrm{Spec}_{(M,\langle\cdot,\cdot\rangle)} = 0$ *and* $(M,\langle\cdot,\cdot\rangle) \vDash ((\mathrm{CP})$ *or* $(\mathrm{ID}))$ ,

   *then* $h_* : \mathcal{M}in_{(M,\langle\cdot,\cdot\rangle)} \to \mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ *is continuous ;*

4. *if one of the following holds:*
   - $\bigwedge \mathrm{Spec}_{(L,[\cdot,\cdot])} = 0$ , $(L,[\cdot,\cdot]) \vDash ((\mathrm{CP})$ *or* $(\mathrm{ID}))$ *and*

   $(\forall\, a \in \mathrm{Cp}(L))\,(h(a^{\perp}) \in \mathrm{Cp}(M))$ *($\Leftarrow \mathrm{Cp}(L)$ is closed w.r.t. $^{\perp}$ ),*
   - $\bigwedge \mathrm{Spec}_{(M,\langle\cdot,\cdot\rangle)} = 0$ , $(M,\langle\cdot,\cdot\rangle) \vDash ((\mathrm{CP})$ *or* $(\mathrm{ID}))$ *and*

   $(\forall\, a \in \mathrm{Cp}(L))\,(h(a)^{\perp} \in \mathrm{Cp}(M))$ *($\Leftarrow \mathrm{Cp}(M)$ is closed w.r.t. $^{\perp}$ ),*

   *then* $h_* : \mathcal{M}in^{-1}_{(M,\langle\cdot,\cdot\rangle)} \to \mathcal{M}in_{(L,[\cdot,\cdot])}$ *is continuous .*

If, furthermore, $h_* \mid_{\mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)}} : \mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)} \to \mathrm{Min}_{(L, [\cdot, \cdot])}$ is injective, then:

- if $h(L) \supseteq \mathrm{Cp}(M)$, then $h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)} \to \mathcal{M}in_{(L, [\cdot, \cdot])}$ is a homeomorphism;

- if $h(\mathrm{Cp}(L)) = \mathrm{Cp}(M)$, then $h_* : \mathcal{M}in^{-1}_{(M, \langle \cdot, \cdot \rangle)} \to \mathcal{M}in^{-1}_{(L, [\cdot, \cdot])}$ is a homeomorphism;

- if one of the following holds:
  - $\bigwedge \mathrm{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$, $h(\mathrm{Cp}(L)) = \mathrm{Cp}(M)$ and $\mathrm{Cp}(L)$ is closed w.r.t. $^{\perp}$,
  - $\bigwedge \mathrm{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$ and $\{h(a^{\perp}) \mid a \in \mathrm{Cp}(L)\} \supseteq \mathrm{Cp}(M)$,
  - $\bigwedge \mathrm{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$, $(M, \langle \cdot, \cdot \rangle) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$ and $\{h(a)^{\perp} \mid a \in \mathrm{Cp}(L)\} \supseteq \mathrm{Cp}(M)$,

  then $h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)} \to \mathcal{M}in^{-1}_{(L, [\cdot, \cdot])}$ is a homeomorphism;

- if one of the following holds:
  - $\bigwedge \mathrm{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$ and $h(\mathrm{Cp}(L)) = \mathrm{Cp}(M)$,
  - $\bigwedge \mathrm{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \vDash ((\mathrm{CP})$ or $(\mathrm{ID}))$ and $\{h(a^{\perp}) \mid a \in \mathrm{Cp}(L)\} = \mathrm{Cp}(M)$,
  - $\bigwedge \mathrm{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \vDash (\mathrm{ID})$ and $\{h(a^{\perp}) \mid a \in L\} \supseteq \mathrm{Cp}(M) \supseteq \{h(a^{\perp}) \mid a \in \mathrm{Cp}(L)\}$,

- $\bigwedge \mathrm{Spec}_{(M,\langle\cdot,\cdot\rangle)} = 0$ , $(M,\langle\cdot,\cdot\rangle) \vDash ((\mathrm{CP}) \mathrm{or} (\mathrm{ID}))$ and $\{h(a)^{\perp} | a \in \mathrm{Cp}(L)\} = \mathrm{Cp}(M)$ ,

- $\bigwedge \mathrm{Spec}_{(M,\langle\cdot,\cdot\rangle)} = 0$ , $(M,\langle\cdot,\cdot\rangle) \vDash (\mathrm{ID})$ and

  $\{h(a)^{\perp} \mid a \in L\} \supseteq \mathrm{Cp}(M) \supseteq \{h(a)^{\perp} \mid a \in \mathrm{Cp}(L)\}$ ,

then $h_* : \mathcal{M}in^{-1}_{(M,\langle\cdot,\cdot\rangle)} \to \mathcal{M}in_{(L,[\cdot,\cdot])}$ is a homeomorphism .

Let:

- $S \subseteq L$ and $T \subseteq M$

## Definition

We call the map $h$:

- $S, T$–**rigid** iff $(\forall\, b \in T)\,(\exists\, a \in S)\,(h(a)^{\perp} = b^{\perp})$

- $T$–**quasirigid** iff $h$ is $\mathrm{Cp}(L), T$–rigid

- $T$–**weak rigid** iff $h$ is $L, T$–rigid

- an $S, T$–$r$–**map** iff

$$(\forall\, b \in T)\,(\forall\, m \in D_{(M,\langle\cdot,\cdot\rangle)}(b) \cap \mathrm{Min}_{(M,\langle\cdot,\cdot\rangle)})\,(\exists\, a \in S)\,(h(a) \nleq m \text{ and } b^{\perp} \leq h(a)^{\perp})$$

- a $T$–**quasi** $r$–**map** iff $h$ is a $\mathrm{Cp}(L), T$–$r$–map

- a $T$–**weak** $r$–**map** iff $h$ is an $L, T$–$r$–map

- an $S, T$–$r^{*}$–**map** iff

$$(\forall\, b \in T)\,(\forall\, m \in V_{(M,\langle\cdot,\cdot\rangle)}(b) \cap \mathrm{Min}_{(M,\langle\cdot,\cdot\rangle)})\,(\exists\, a \in S)\,(h(a) \leq m \text{ and } h(a)^{\perp} \leq b^{\perp})$$

- a $T$–**quasi** $r^{*}$–**map** iff $h$ is a $\mathrm{Cp}(L), T$–$r^{*}$–map

- a $T$–**weak** $r^{*}$–**map** iff $h$ is an $L, T$–$r^{*}$–map

We call the morphism $f$:

- **rigid**, **quasirigid**, respectively **weak rigid** iff the map $f^\bullet$ is $\mathrm{PCon}(A), \mathrm{PCon}(B)$–rigid, $\mathrm{PCon}(B)$–quasirigid, respectively $\mathrm{PCon}(B)$–weak rigid;

- an $r$, **quasi $r$**, respectively **weak $r$–morphism** iff $f^\bullet$ is a $\mathrm{PCon}(A), \mathrm{PCon}(B)$–$r$, $\mathrm{PCon}(B)$–quasi $r$, respectively $\mathrm{PCon}(B)$–weak $r$–map;

- an $r^*$, **quasi $r^*$**, respectively **weak $r^*$–morphism** iff $f^\bullet$ is a $\mathrm{PCon}(A), \mathrm{PCon}(B)$–$r^*$, $\mathrm{PCon}(B)$–quasi $r^*$, respectively $\mathrm{PCon}(B)$–weak $r^*$–map.

If $A \hookrightarrow B$, then $A \hookrightarrow B$ is:

- a **rigid**, **quasirigid**, respectively **weak rigid extension** iff the morphism $i_{A,B}$ is rigid, quasirigid, respectively weak rigid;

- an $r$, **quasi $r$**, respectively **weak $r$–extension** iff $i_{A,B}$ is an $r$, quasi $r$, respectively weak $r$–morphism;

- an $r^*$, **quasi $r^*$**, respectively **weak $r^*$–extension** iff $i_{A,B}$ is an $r^*$, quasi $r^*$, respectively weak $r^*$–morphism.

Let:

- $S_1 \subseteq S_2 \subseteq L$ and $T_1 \subseteq T_2 \subseteq M$

## Proposition

- If $S$ is closed w.r.t. arbitrary $\bigvee$ and $T_1$ is $\vee$–dense in $T_2$, then:

  $h$ is $S, T_1$–rigid iff $h$ is $S, T_2$–rigid ;

  $h$ is an $S, T_1$–$r^*$–map iff $h$ is an $S, T_2$–$r^*$–map .

- Thus:

  $f$ is weak rigid iff $f^\bullet$ is $\mathrm{Cp}(\mathrm{Con}(B))$–weak rigid iff $f^\bullet$ is $\mathrm{Con}(B)$–weak rigid ;

  $f$ is a weak $r^*$–morphism iff $f^\bullet$ is a $\mathrm{Cp}(\mathrm{Con}(B))$–weak $r^*$–map

  iff $f^\bullet$ is a $\mathrm{Con}(B)$–weak $r^*$–map .

## Proposition

- If $S_1$ is $\vee$–dense in $S_2$ and $T_1$ is $\vee$–dense in $T_2$, then:

  $h$ is an $S_2, T_1$–$r$–map iff $h$ is an $S_1, T_2$–$r$–map . Thus:

- $f$ is an $r$–morphism iff $f$ is a quasi $r$–morphism iff $f$ is a weak $r$–morphism

  iff $f^\bullet$ is a $\mathrm{PCon}(A), \mathrm{Con}(B)$–$r$–map iff $f^\bullet$ is a $\mathrm{Con}(B)$–quasi $r$–map

  iff $f^\bullet$ is a $\mathrm{Con}(B)$–weak $r$–map .

## Proposition

Assume that $\bigwedge \operatorname{Spec}_{(M,\langle\cdot,\cdot\rangle)}(0) = 0$ and $h_*(\operatorname{Min}_{(M,\langle\cdot,\cdot\rangle)}) \subseteq \operatorname{Min}_{(L,[\cdot,\cdot])}$ .

If $h$ is $S, T$–rigid and one of the following holds:

- $(M, \langle\cdot,\cdot\rangle) \vDash (\mathrm{ID})$ ,

- $h(S) \subseteq \operatorname{Cp}(M)$ and $(M, \langle\cdot,\cdot\rangle) \vDash (\mathrm{CP})$ ,

then $h$ is both an $S, T$–$r$–map and an $S, T$–$r^*$–map .

## Proposition

Assume that $L$ is algebraic , $1 \in \operatorname{Cp}(M)$ , $h_*(\operatorname{Spec}_{(M,\langle\cdot,\cdot\rangle)}) \subseteq \operatorname{Spec}_{(L,[\cdot,\cdot])}$ ,

$h(\operatorname{Cp}(L)) \subseteq \operatorname{Cp}(M)$ , $h \vDash [,]_{\leq} \operatorname{Cp}(L)$ and $T$ is $\vee$–dense in $M$ .

If $h$ is an $S, T$–$r$–map or an $S, T$–$r^*$–map and one of the following holds:

- $M$ is algebraic and $(M, \langle\cdot,\cdot\rangle) \vDash (\mathrm{ID})$ ,

- $h(S) \cup T \subseteq \operatorname{Cp}(M)$ and $(M, \langle\cdot,\cdot\rangle) \vDash (\mathrm{CP})$ ,

then $h$ is $\operatorname{Min}$–admissible and

$h_* |_{\operatorname{Min}_{(M,\langle\cdot,\cdot\rangle)}} \colon \operatorname{Min}_{(M,\langle\cdot,\cdot\rangle)} \to \operatorname{Min}_{(L,[\cdot,\cdot])}$ is a bijection .

## Theorem

*Assume that* $L$ *and* $M$ *are algebraic*, $1 \in \mathrm{Cp}(M)$, $T$ *is* $\vee$*-dense in* $M$,

$\bigwedge \mathrm{Spec}_{(M, \langle \cdot, \cdot \rangle)}(0) = 0$, $h_*(\mathrm{Spec}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Spec}_{(L, [\cdot, \cdot])}$, $h(\mathrm{Cp}(L)) \subseteq \mathrm{Cp}(M)$,

$h \vDash [,]_{\leq} \mathrm{Cp}(L)$ *and one of the following holds:*

- $M$ *is algebraic and* $(M, \langle \cdot, \cdot \rangle) \vDash (\mathrm{ID})$,
- $h(S) \cup T \subseteq \mathrm{Cp}(M)$ *and* $(M, \langle \cdot, \cdot \rangle) \vDash (\mathrm{CP})$.

*Let us consider the following statements:*

⓪ $h$ *is* $S, T$*-rigid*; ① $h$ *is an* $S, T$*-r-map*; ② $h$ *is an* $S, T$*-r\*-map*;

③ $h_*(\mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Min}_{(L, [\cdot, \cdot])}$ *and*

$h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)} \to \mathcal{M}in_{(L, [\cdot, \cdot])}$ *is a homeomorphism*;

④ $h_*(\mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Min}_{(L, [\cdot, \cdot])}$ *and*

$h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)}^{-1} \to \mathcal{M}in_{(L, [\cdot, \cdot])}^{-1}$ *is a homeomorphism*;

⑤ $h_*(\mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Min}_{(L, [\cdot, \cdot])}$ *and*

$h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)} \to \mathcal{M}in_{(L, [\cdot, \cdot])}$ *is bijective and open*;

⑥ $h_*(\mathrm{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \mathrm{Min}_{(L, [\cdot, \cdot])}$ *and*

$h_* : \mathcal{M}in_{(M, \langle \cdot, \cdot \rangle)}^{-1} \to \mathcal{M}in_{(L, [\cdot, \cdot])}^{-1}$ *is bijective and open*.

Then:

- ③⇒⑤ , ④⇒⑥ , ① ⇒ (③ and ⑤) ;

- (① or ②) ⇒ ((③⇔⑤) and (④⇔⑥)) ;

- ⓪ ⇒ (①, ②, ③ and ⑤) ;

- if $S \subseteq \mathrm{Cp}(L)$ and $T \supseteq \mathrm{Cp}(M)$ , then ⓪ ⇒ (④ and ⑥) ;

- if $S$ is ∨–dense in $L$ , then: ①⇔③⇔⑤⇔ $h$ is $L, M$–rigid ;

- if $S = \mathrm{Cp}(L)$ and $T = \mathrm{Cp}(M)$ , then ①⇔②⇔③⇔④⇔⑤⇔⑥ ;

- if $L = \mathrm{Cp}(L)$ and $M = \mathrm{Cp}(M)$ , then ①⇔②⇔③⇔④⇔⑤⇔⑥ .

THANK YOU FOR YOUR ATTENTION !