Local finiteness in varieties of MS4-algebras

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Joint work with Guram Bezhanishvili (NMSU)

Outline

An S4-algebra is a tuple $\mathfrak{A} = (B, \Diamond)$ such that

- B is a Boolean algebra (with $\wedge, \vee, -, 0, 1$)
- $\Diamond : B \to B$ is a unary operation satisfying the identities of a closure operator:
  - $\Diamond 0 = 0$
  - $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
  - $a \leq \Diamond a$
  - $\Diamond\Diamond a \leq a$
- We write $\Box := -\Diamond-$, which is an interior operator.

S4-algebras provide algebraic semantics for the modal logic S4, which is the smallest normal modal logic in one modality $\Diamond$ containing

- $\Diamond\bot \leftrightarrow \bot$
- $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
- $p \to \Diamond p$
- $\Diamond\Diamond p \to p$

Note: We will conflate formulas with terms

- $\mathfrak{A} \models \varphi$ means $\mathfrak{A} \models (\varphi = 1)$
- $\mathfrak{A} \models \varphi \to \psi$ iff $\mathfrak{A} \models \varphi \leq \psi$

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If $\mathfrak{A} = (B, \Diamond)$ is an S4-algebra, we write $H_\Box = \Box B$ for the set of $\Box$-fixpoints. In fact,

- $H_\Box$ is a bounded sublattice of B that forms a Heyting algebra, where
- Heyting implication is given by $a \to b = \Box(-a \vee b)$.
- This is intimately related to the Gödel translation of intuitionistic propositional logic (IPC) into the modal logic S4.

Outline

From any **set** X, we get a Boolean algebra $\mathcal{P}(X)$.

Conversely, Stone duality gives us a representation theorem:

- Every Boolean algebra is representable as a subalgebra of $\mathcal{P}(X)$ for some set X.
- We may identify the precise subalgebra as the **clopen** subsets in a certain **Stone topology** on X.

The duality is implemented as follows:

- From B, let $X = \mathrm{Uf}\, B$; the map $a \mapsto \mathfrak{s}(a) = \{x : a \in x\}$ is an embedding $B \hookrightarrow \mathcal{P}(X)$.
- In fact, $\{\mathfrak{s}(a) : a \in B\}$ is a clopen basis for a zero-dimensional compact Hausdorff (Stone) topology on X, and B is isomorphic to the subalgebra of clopen subets of X.

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From any relational structure (X, R) where R is a transitive reflexive relation (quasiorder) on X (an **S4-frame**), we get an S4-algebra $(\mathcal{P}(X), \Diamond_R)$ where

$$\Diamond_R(U) = R^{-1}(U) = \{x \in X : xRy \text{ for some } y \in U\}$$

Conversely, Jónsson–Tarski duality gives a representation theorem:

- Every S4-algebra is representable as a subalgebra of (X, R) for some S4-frame (X, R).
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- From $\mathfrak{A} = (B, \Diamond)$, let $X = \mathrm{Uf}\, B$ and define xRy iff $x \cap H_\Box \subseteq y$, the map $a \mapsto \mathfrak{s}(a) = \{x : a \in x\}$ is an embedding $\mathfrak{A} \hookrightarrow (\mathcal{P}(X), R_\Diamond)$.
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The relation $R_\Diamond$ will turn out to be 'compatible' with the topology on X, in that it satisfies:

- $R(x) = \{y \in X : xRy\}$ is closed
- $R^{-1}(U)$ is clopen whenever U is clopen.

A relation on a Stone space satisfying these properties is said to be *continuous*.

We say that (X, R) is a *descriptive* S4-*frame* when X is a Stone space and R is a continuous quasiorder on X.

In the same way that Stone duality ultimately yeilds a dual categorical equivalence between

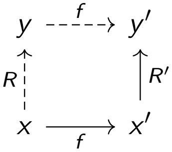
- The category **Bool** of Boolean algebras, with homomorphisms
- The category **Stone** of Stone spaces (zero-dimensional compact Hausdorff spaces), with continuous maps

We have a dual equivalence between

- The category **S4** of S4-algebras, with homomorphisms
- The category **DS4Fr** of descriptive S4-frames, with continuous *p-morphisms* or *bounded morphisms*.

A map $f : (X, R) \to (X', R')$ is p-morphic if it preserves and reflects the relation, e.g.

- xRy implies $f(x)\, R'\, f(y)$
- $f(x)\, R'\, y'$ implies xRy for some y with $f(y) = y'$.



Given $\mathfrak{A} = (B, \Diamond)$, the *dual frame* or *dual space* of $\mathfrak{A}$ is the descriptive frame $\mathfrak{F} = (X, R)$.

- Subalgebras of $\mathfrak{A}$ correspond to *quotients* of $\mathfrak{F}$ by *correct partitions*, which are 'compatible' equivalence relations = kernels of continuous p-morphsisms.
- Quotients or homomomorphic images of B correspond to closed *generated subframes* of $\mathfrak{F}$ (subframes closed under R).

We also have dual characterizations of simplicity and subdirect irreducibility:

- $x \in X$ is a *root* if $R(x) = X$, $\mathfrak{F}$ is strongly rooted if it has an open set of roots.
- $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{F}$ is strongly rooted.
- $\mathfrak{A}$ is simple iff $\mathfrak{F}$ iff every point of X is a root (X is an R-cluster).

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For an S4-algebra $\mathfrak{A}$, define the *depth* of $\mathfrak{A}$ to be the longest length of a proper R-chain in $\mathfrak{F}$. If there is no longest length, the depth is ω.

A variety $\mathbf{V} \subseteq \mathbf{S4}$ has *depth* $\leq n$ if the depth of any algebra $\mathfrak{A} \in \mathbf{V}$ is bounded by n. If $\mathbf{V}$ contains algebra of arbitrary depth, then it is of depth ω.

Depth is controlled by the well-known identities:

$$P_1 = \Diamond\Box q_1 \to \Box q_1 \qquad P_n = \Diamond(\Box q_n \to \neg P_{n-1}) \to \Box q_n$$

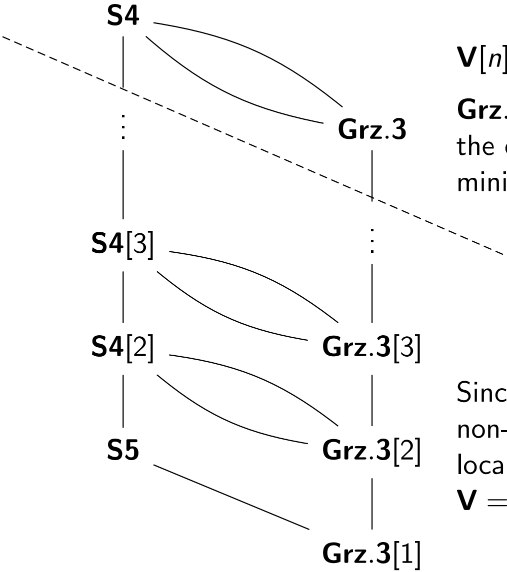
Theorem

$\mathbf{V} \models P_n$ *iff the depth of* $\mathbf{V}$ *is* $\leq n$

Theorem (Segerberg–Maksimova)

$\mathbf{V} \subseteq \mathbf{S4}$ *is locally finite iff* $\mathbf{V} \models P_n$ *for some* n.

The subvariety lattice of **S4**:



$\mathbf{V}[n] = \mathbf{V} + P_n$, $\mathbf{S5} = \mathbf{S4}[1]$

Grz.**3**$[n]$ is the variety generated by the chain of length n, and is the minimal variety of depth n.

Since **Grz**.**3** is the unique minimal non-locally-finite subvariety of **S4**, local finiteness is decidable (given $\mathbf{V} = \mathbf{S4} + \varphi$, decide if $\mathbf{V} \supseteq \mathbf{Grz.3}$)

Outline

An *monadic* S4-*algebra* or MS4-algebra is a tuple $\mathfrak{A} = (B, \Diamond, \exists)$ such that

- B is a Boolean algebra (with $\wedge, \vee, -, 0, 1$)
- $\Diamond$ is an S4-operator (a closure operator)
- $\exists$ is an S5-operator (an S4-operator satisfying $P_1 := \exists\forall a \leq \forall a$)
- $\exists\Diamond a \leq \Diamond\exists a$
- We write $\Box := -\Diamond-$ and $\forall = -\exists-$

MS4-algebras provide semantics for the one-variable fragment of the standard predicate-logic extension of S4 propositional logic (QS4). The translation is, e.g.

$$\exists\Diamond p \to \Diamond\exists p \quad \stackrel{T}{\longmapsto} \quad \exists x\, \Diamond p(x) \to \Diamond\exists x\, p(x)$$

and φ is a theorem of MS4 iff $T(\varphi)$ is a theorem of QS4.

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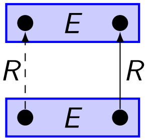
A *descriptive* MS4-*frame* is a tuple (X, R, E) where

- X is Stone space
- R is a continuous quasi-order on X
- E is a continuous equivalence relation on X
- $RE \subseteq ER$



Jónsson–Tarski duality again yields a dual equivalence between

- The category **MS4** of MS4-algebras, with homomorphisms
- The category **DMS4Fr** of descriptive MS4-frames with continuous p-morphisms
    - Such morphisms are p-morphic with respect to both R and E.

Thus every MS4-algebra $(B, \Diamond, \exists)$ can be thought of as the algebra of clopen subsets of (X, R, E), where

$$\Diamond U = R^{-1}(U) \qquad \exists U = E(U)$$

For any $\mathfrak{A} = (B, \Diamond, \exists)$, let

- $B_0 = \exists B$ be the set of $\exists$-fixpoints
- $\Diamond_0 = \Diamond|_{B_0}$ the restriction of $\Diamond$

Then $\mathfrak{A}_0 = (B_0, \Diamond_0)$ an S4-subalgebra of $(B, \Diamond)$. This follows from the fact that $\exists\Diamond \leq \Diamond\exists$.

Dually, for any (X, R, E), let

- $X_0 = X/E$
- R_0 defined by $\alpha\, R_0\, \beta$ iff $\forall x \in \alpha, \exists y \in \beta, xRy$.

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- In an algebra $\mathfrak{A} = (B, \Diamond, \exists)$, define $\blacklozenge = \Diamond\exists$
- In a descriptive frame $\mathfrak{F} = (X, R, E)$, define $Q = ER$

We may act as if $\blacklozenge$ is an S4-operator/closure operator on B that corresponds dually to the quasi-order Q. We have the dual operator $\blacksquare = -\blacklozenge-$.

This is the appropriate notion to get an analogous characterization of simplicity and subdirect irreducibility:

- $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{F}$ is strongly Q-rooted
- $\mathfrak{A}$ is simple iff $\mathfrak{F}$ every point of X is a Q-root (X is a Q-cluster).

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A variety $\mathbf{V}$ is *semisimple* if every subdirectly irreducible algebra is simple.

The previous characterization yields that $\mathbf{V} \subseteq \mathbf{MS4}$ is semisimple iff any of the following equivalent conditions hold

- any Q-rooted dual frame from $\mathbf{V}$ is a Q-cluster
- In any dual frame from $\mathbf{V}$, Q is an equivalence relation
- $\mathbf{V}$ has Q-depth 1, e.g. $\mathbf{V} \models P_1^{\blacklozenge} := \blacksquare\blacklozenge p \leq \blacklozenge p$.

Theorem

$\mathbf{MS4_S} := \mathbf{MS4} + (\blacksquare\blacklozenge a \leq \blacklozenge a)$ *is the largest semisimple subvariety of* $\mathbf{MS4}$.

Theorem (Bezhanishvili, M.)

$\mathbf{MS4_S}$ *is finitely approximable (generated by its finite members), and so the logic* $\mathrm{MS4_S}$ *has the finite model property and is decidable.*

This generalizes the analogous situation for two well-known logics that are actually extensions of the logic $\mathrm{MS4_S}$.

- $\mathrm{S4}_u$ – S4 augmented with a *universal modality*)
- $\mathrm{S5}^2$ – the *product* of the modal logic S5 with itself.

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We will speak of the *depth* of an MS4-algebra $(B, \Diamond, \exists)$ or a variety of MS4-algebras always meaning the depth of the S4-reduct $(B, \Diamond)$ or, equivalently, the R-depth of the dual frame.

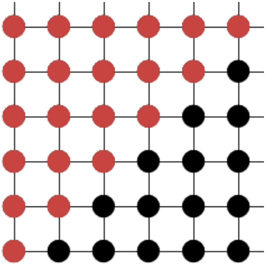
A direct analogue of the Segerberg–Maksimova theorem does not hold; already $\mathbf{MS4}[1] = \mathbf{S5}^2$ is not locally finite.

This was first noticed by Henkin, Monk and Tarski in their study of cylindric algebras.

- $\mathbf{S5}^2$ = the variety of *two-dimensional diagonal-free cylindric algebras*.

In $S5^2$-frames, the relations are two commuting equivalence relations (X, E_1, E_2).

Henkin–Monk–Tarski give a one-generated infinite subalgebra of the $(\mathcal{P}(\omega \times \omega), E_1, E_2)$ where the equivalence classes of E_1 and E_2 are rows and columns.



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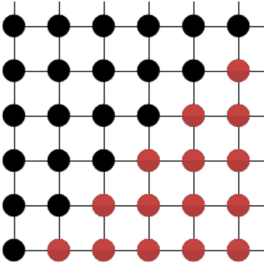

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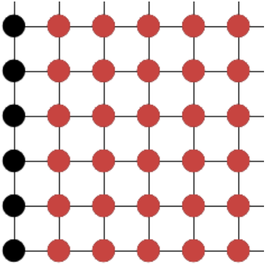

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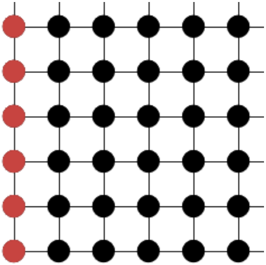

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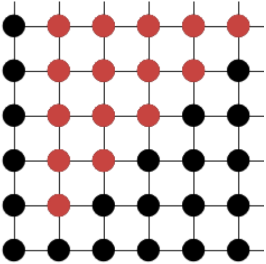

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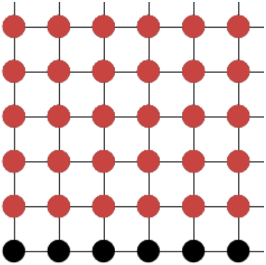

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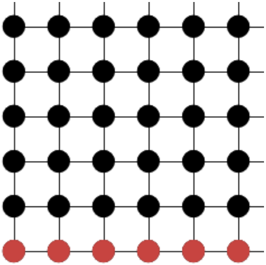

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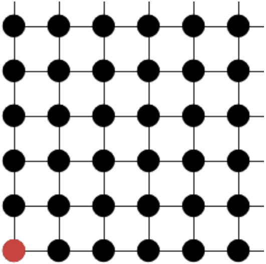
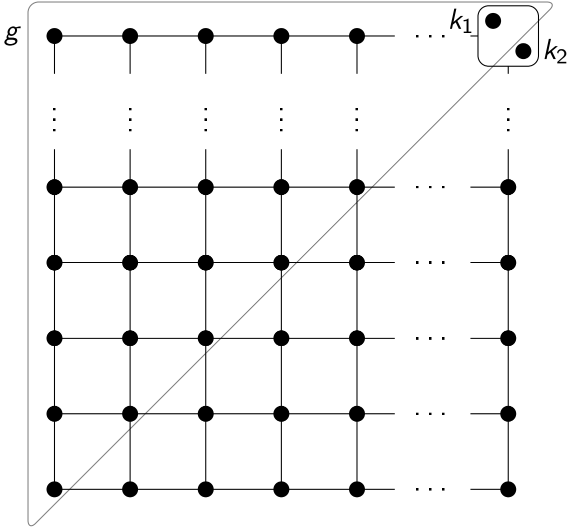


We characterize the dual frame of this algebra, in order to give an explicit example of an MS4-algebra and descriptive MS4-frame that is one-generated and infinite.



The *layers* of a quasiorder can be defined as

$$D_1 = \max X \qquad D_{n+1} = \max\left(X - \bigcup_{i=1}^{n} D_i\right)$$

For descriptive S4-frames,

1. D_1 is always closed.
2. If $\mathfrak{F}$ is finitely generated, then every finite layer is clopen, and consists of isolated points.

This example shows that finitely generated MS4-algebras can have limit points of finite depth; we do not know if it remains true that the layers themselves are clopen in this case.

We would also like to know if there is an MS4-frame whose layers *only* consist of limit points.

Since *every* proper subvariety of $\mathbf{S5}^2$ is locally finite, and every layer of an MS4-algebra is an $\mathrm{S5}^2$-frame, this suggests the reasonable conjecture that a variety of MS4-algebras is locally finite if

1. It is finite depth (necessary)
2. Every layer is locally finite as an $\mathrm{S5}^2$-frame ("finite width")

This conjecture fails drastically in general, but first some positive results.

We give the following opaque characterization of local finiteness:

Theorem (Bezhanishvili, M.)

$\mathbf{V} \subseteq \mathbf{MS4}$ *is locally finite iff*

1. *It is finite depth* ($\mathbf{V} \models P_n$ *for some* n)
2. *There is a uniform bound* $f : \mathbb{N} \to \mathbb{N}$ *such that, for every n-generated s.i.* $\mathfrak{A} \in \mathbf{V}$*, the algebra* $\mathfrak{A}_0$ *is* $f(n)$*-generated as an S4-algebra.*

Using this criterion, we obtain

Theorem

A variety $\mathbf{V} \subseteq \mathbf{MS4} + \mathrm{alt}_k^0$ *is locally finite iff it is finite depth.*

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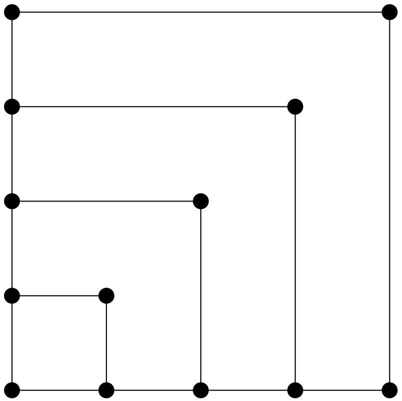
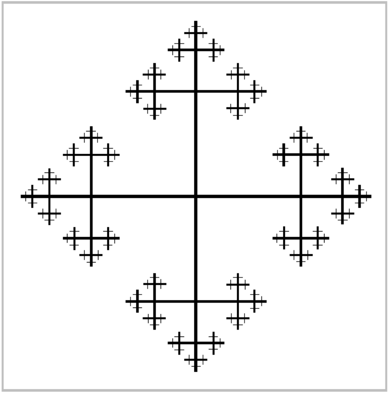
S5$_2$ is the variety of S5$_2$-algebras: tuples $(B, \exists_1, \exists_2)$ where the $\exists_i$ are **unrelated** S5-operators.

- Dually, these correspond to frames (X, E_1, E_2) where the E_i are **unrelated** equivalence relations.
- S5^2 is the *product* of S5 with itself (= commutivity axiom)
- S5$_2$ is the *fusion* of S5 with itself (= no axioms relating the modalities)





A characterization of local finiteness in $\mathbf{S5}_2$ is wide-open.

- Fusions do not preserve local finiteness
- In general, only highly restrictive sufficient conditions are known to guarantee a fusion to be locally finite.

Let $\Lambda(\mathbf{V})$ denote the lattice of subvarieties of a variety $\mathbf{V}$.

Theorem (Bezhanishvili, M.)

There is a translation $T : \Lambda(\mathbf{S5}_2) \to \Lambda(\mathbf{MS4}_\mathbf{S}[2])$ that preserves and reflects local finiteness (that is, $\mathbf{V}$ is locally finite iff $T(\mathbf{V})$ is locally finite)

In particular, characterizing local finiteness even in $\mathbf{MS4}_\mathbf{S}[2]$ is as hard as the corresponding problem for $\mathbf{S5}_2$.

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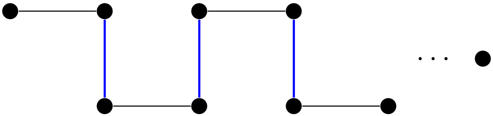
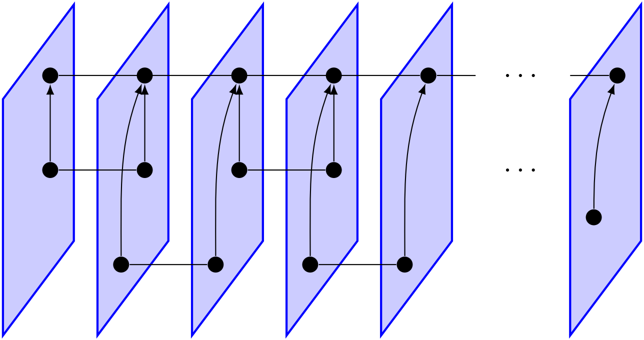
Proof in one slide



$\mathfrak{F}$

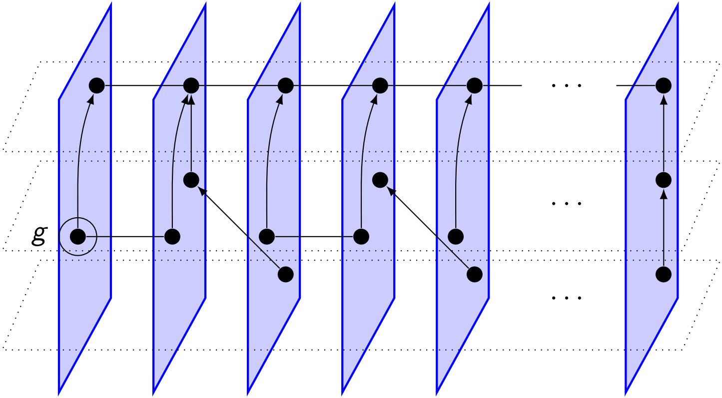


$T(\mathfrak{F})$

Some other consequences:

- MS4 is the *expanding relativized product* (or *semiproduct*) of S4 and S5 (impose left commutivity axiom $\exists\Diamond \to \Diamond\exists$).
- Our construction actually extends to the full *product*, where one imposes the full commutivity condition $\exists\Diamond \leftrightarrow \Diamond\exists$.
- By adding a "bottom rail" to the translation, one can ensure that the relations R and E actually commute in the resulting frame.

We also provide an example demonstrating that local finiteness of layers (as $S5_2$-algebras) is not even sufficient for local finiteness already in depth-3:



Outline

The following formulas are well-known in the study of intuitionistic predicate logic

$$\mathsf{Cas} := \forall x((P(x) \to \forall y P(y)) \to \forall y P(y)) \to \forall x P(x)$$

$$\mathsf{K} := \forall x \neg\neg P(x) \to \neg\neg \forall x P(x)$$

Both give rise to distinct intermediate predicate logics, and Cas is strictly weaker than K.

Cas plays a role in extending the translations

$$\mathsf{IPC} \xrightarrow{\text{Gödel translation}} \mathsf{Grz} \xrightarrow{\text{splitting translation}} \mathsf{GL}$$

where

- IPC = Intuitionistic propositional logic
- Grz = S4 + grz is the Grzegorczyk logic
- GL = K4 + Löb's axiom = the logic of the Peano Arithmetic provability predicate.

Let

- $\mathsf{MCas} = \forall((p \to \forall p) \to \forall p) \to \forall p$
  - the monadic version of the Casari formula
- $\mathsf{MCas}^t = \blacksquare(\Box(\Box p \to \blacksquare p) \to \blacksquare p) \to \blacksquare p$
  - the Gödel translation of MCas
  - $(\blacksquare = \Box\forall)$

Bezhanishvili, Brantley and Ilin showed that these formulas are exactly what is required to lift the previous translations to the monadic setting:

$$\underbrace{\mathsf{MIPC} + \mathsf{MCas}}_{\mathsf{M^+IPC}} \xrightarrow{\text{Gödel translation}} \underbrace{\mathsf{MGrz} + \mathsf{MCas}^t}_{\mathsf{M^+Grz}} \xrightarrow{\text{splitting translation}} \mathsf{MGL}$$

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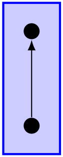
It is natural then to examine the variety $\mathsf{M^+S4} = \mathsf{MS4} + \mathsf{MCas}^t$.

Roughly, $\mathsf{M^+S4}$-frames do not have *dirty clusters*, an E-class containing properly R-related points.



Moreover,

Theorem (M.)

- *In an* $\mathsf{M^+S4}$*-frame* $\mathfrak{F}$*, every layer of finite depth is* E*-saturated, and hence every layer is a (refined)* $\mathsf{S5}^2$*-frame.*
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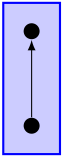

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Finally, the local finiteness situation already seems more under control:

Theorem (M.)

$\mathbf{V} \subseteq \mathbf{M^+S4}$ *is locally finite iff*

1. $\mathbf{V}$ *is of finite depth (decidable)*
2. $\mathbf{V}_T$ *is locally finite – equivalently,* $\mathbf{V}_T \subset \mathbf{S5}^2$.

Here $\mathbf{V}_T$ is a subvariety of $\mathbf{S5}^2$ defined semantically from the layers of algebras in $\mathbf{V}$; work is ongoing to investigate the decidability of (2).

Thanks!