Higher-dimensional book-spaces

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Outline

In this talk, I'll show that, for each $n \geq 2$, there exists an infinite family of compact n-dimensional simplicial complexes which admit the structure of topological modular lattice but not topological distributive lattice.

Outline

- Topological lattices
- Geometric realization
- Taylor's book-space result
- Higher dimensions

Topological lattices

- A *topological lattice* is a model of the theory of lattices in Top, the category of topological spaces.
- Examples include the real numbers $\mathbb{R}$ with the usual order, an interval therein like $[0, 1]$, or any lattice equipped with the discrete topology.

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Topological lattices

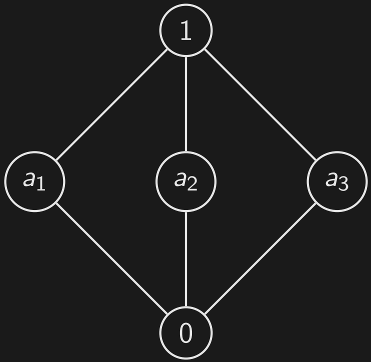
More concretely, a topological lattice is a structure $(L, \wedge, \vee, \tau)$ such that

1. τ is a topology on L,
2. $(L, \wedge, \vee)$ is a lattice, and
3. $\wedge$ and $\vee$ are both continuous maps from L^2 to L, where L^2 carries the product topology induced by τ.

Geometric realization

- We would like a way to produce more examples of topological lattices, beyond just taking products, subalgebras, etc. in the appropriate category.
- Geometric realization does that for us.

Geometric realization



- Let L be the lattice pictured above.

Geometric realization

- The *s-level set* of $f\colon L \to [0,1]$ is

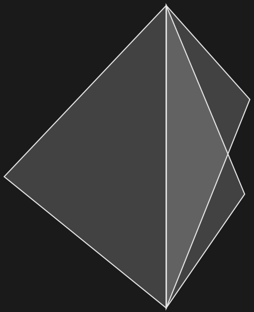
$$f_s = \{\, x \in L \mid f(x) \geq s \,\}\,.$$

- A function $f\colon L \to [0,1]$ as *L-admissible* when f_s is a principal ideal of L for each $s \in [0,1]$.

Geometric realization

- We define the *geometric realization* $\Gamma(L)$ of L to be the subspace of $[0,1]^L$ consisting of all L-admissible functions $f: L \to [0,1]$.
- The order on $[0,1]^L$ restricts to a lattice order on $\Gamma(L)$ for which $\wedge$ and $\vee$ are continuous.

Geometric realization



- The space $\Gamma(L)$ is pictured above.

Geometric realization

- We can play the same game with any (finite) lattice L.
- We always get the same space as the geometric realization of the order complex of L, which is the "usual" geometric realization of L from topological combinatorics.

Taylor's book-space result

- In 2017, Taylor gave a family of 2-dimensional compact simplicial complexes which carried the structure of topological modular lattice but not topological distributive lattice.
- He asked whether higher-dimensional examples exist, and the answer is yes.

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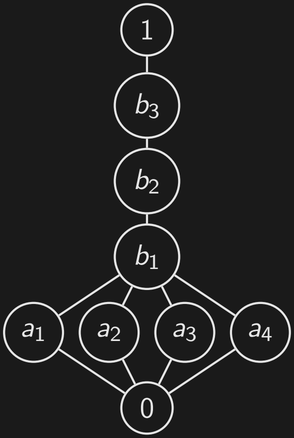
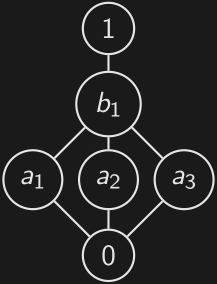
Higher dimensions



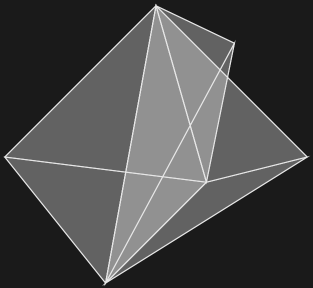
- Consider the *book lattices* $M_{d,n}$, with $M_{5,4}$ pictured above.

Higher dimensions



- These lattices are all modular, but not distributive if $n > 2$.

Higher dimensions



- The geometric relatization $\Gamma(M_{d,n})$ is a compact d-dimensional simplicial complex.

Higher dimensions

Porism

When $n \geq 3$ the book-space $\Gamma(M_{d,n})$ does not embed in $\mathbb{R}^d$.

Higher dimensions

Theorem (A. ~~2018~~ 2024)

The spaces $\Gamma(M_{d,n})$ admit a structure of modular lattice but not distributive lattice when $n \geq 3$.

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The spaces $\Gamma(M_{d,n})$ admit a structure of modular lattice but not distributive lattice when $n \geq 3$.

- The lattices $\Gamma(M_{d,n})$ are subdirect powers of the corresponding lattices $M_{d,n}$ via the maps $f \mapsto \bigvee f_s$ for each $s \in [0,1]$.
- This means that the $\Gamma(M_{d,n})$ are modular, since the $M_{d,n}$ are modular.

Higher dimensions

Theorem (A. ~~2018~~ 2024)

The spaces $\Gamma(M_{d,n})$ admit a structure of modular lattice but not distributive lattice when $n \geq 3$.

- The lattices $\Gamma(M_{d,n})$ are not distributive because they contain a copy of $M_{d,n}$.
- What's more, no continuous lattice operations on the space $\Gamma(M_{d,n})$ can make it into a topological distributive lattice.

Higher dimensions

Theorem (A. ~~2018~~ 2024)

The spaces $\Gamma(M_{d,n})$ admit a structure of modular lattice but not distributive lattice when $n \geq 3$.

- Taylor had used that any d-dimensional simplicial complex with topological distributive lattice structure must embed in $\mathbb{R}^d$, just for the case $d = 2$.
- Our porism said that $\Gamma(M_{d,n})$ does not embed into $\mathbb{R}^d$, so no choice of continuous lattice operations on it can be distributive.

References

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