Model theoretical tameness and CSPs

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Definition

$\mathfrak{B}$: structure with a finite relational signature.
CSP($\mathfrak{B}$) is the following decision problem.

Constraint Satisfaction Problems

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CSP($\mathfrak{B}$) is the following decision problem.

- INPUT: a finite structure $\mathfrak{A}$ (with the same signature as $\mathfrak{B}$)
- QUESTION: Is there a homomorphism $\mathfrak{A} \to \mathfrak{B}$?

Polymorphisms

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$$f : \left(\begin{array}{cccc} \boxed{\begin{matrix} a_{11} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ \vdots \\ a_{n2} \end{matrix}} & \begin{matrix} \dots \\ \\ \dots \end{matrix} & \boxed{\begin{matrix} a_{1k} \\ \vdots \\ a_{nk} \end{matrix}} \end{array} \right) \mapsto \left(\boxed{\begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix}} \right)$$

$$\begin{array}{ccccccc} \Cap & \Cap & & \Cap & & \Cap \\ R & R & \dots & R & \Rightarrow & R \end{array}$$

Notation: $\mathrm{Pol}(\mathfrak{B})$.

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& R & R & \dots & R & \Rightarrow & R
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Theorem (Jeavons '98 + Geiger; Bodnarčuk, Kalužnin, Kotov, Romov '69)

If $\mathfrak{B}$ is finite, then the complexity of $\mathrm{CSP}(\mathfrak{B})$ is uniquely determined by $\mathrm{Pol}(\mathfrak{B})$.

Fact

$\mathrm{Pol}(\mathfrak{A})$ *forms a clone.*

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Definition

$\mathcal{C} \subseteq \bigcup_{k=1}^{\infty} X^{X^k}$ is a clone if

1. $\mathcal{C}$ contains all projections $(\pi_i : (x_1, \ldots, x_k) \mapsto x_i)$
2. $f, g_1, \ldots, g_k \in \mathcal{C} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{C}$.

Definition

$\mathcal{C}, \mathcal{D}$: clones.

$\xi : \mathcal{C} \to \mathcal{D}$ is a clone homomorphism if

1. ξ preserves arities,
2. $\xi(\pi_i) = \pi_i$,
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$\mathfrak{A}$: finite.
$\mathscr{P}$: the clone of projections on a 2-element set.
Facts:

- If $\mathrm{Pol}(\mathfrak{A}) = \mathscr{P}$, then $\mathrm{CSP}(\mathfrak{A})$ is **NP**-complete.

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- $\exists\, \mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ minion homomorphism, then $\mathrm{CSP}(\mathfrak{A})$ is at least as hard as $\mathrm{CSP}(\mathfrak{B})$.

Constraint Satisfaction Problems

Source of hardness: finite case

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Observation

Every finite structure is homomorphically equivalent to a core.

CSP dichotomy

Finite case

Theorem (Barto, Kozik, Siggers, ...)

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If 1 - 4 hold then $\mathrm{CSP}(\mathfrak{A})$ *is in* **P**.

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Therefore if $\mathfrak{A}$ is finite then $\mathrm{CSP}(\mathfrak{A})$ is in **P** or it is **NP**-complete.

Generalizations to infinite structure

ω-categoricity

Disclaimer: every structure is assumed to be countable!

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Definition (the useful one)

$\mathfrak{A}$ is ω-categorical if $\mathrm{Aut}(\mathfrak{A})$ has finitely many n-orbits for all $n \in \omega$.

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This includes all finite structures!

Theorem (Bodirsky, Nešetřil '03)

If $\mathfrak{A}$ is ω-categorical, then the complexity of $\mathrm{CSP}(\mathfrak{A})$ is uniquely determined by $\mathrm{Pol}(\mathfrak{A})$.

Generalizations to infinite structure

Model-complete cores

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Remark: in general we have $\overline{\mathrm{Aut}(\mathfrak{A})} \subseteq \mathrm{Emb}(\mathfrak{A}) \subseteq \mathrm{End}(\mathfrak{A})$.

Theorem (Bodirsky '05)

Every ω-categorical structure is homomorphically equivalent to a model-complete core.
This is a unique up to isomorphism, and again ω-categorical.

$\mathscr{P}$: the clone of projections on a 2-element set.
Facts:

- If $\mathrm{Pol}(\mathfrak{A}) = \mathscr{P}$, then $\mathrm{CSP}(\mathfrak{A})$ is **NP**-complete.
- $\exists\, \mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ uniformly continuous minion homomorphism, then $\mathrm{CSP}(\mathfrak{A})$ is at least as hard as $\mathrm{CSP}(\mathfrak{B})$.
- $\exists\, \mathrm{Pol}(\mathfrak{A}) \to \mathscr{P}$ uniformly continuous minion homomorphism, then $\mathrm{CSP}(\mathfrak{A})$ is **NP**-hard.

Infinite-domain CSP dichotomy

Algebraic formulation

$\mathfrak{A}$ *is an* ω*-categorical* *model-complete core. Then TFAE.*

1. $\nexists\, \mathrm{Pol}(\mathfrak{A}) \to \mathscr{P}$ *minion homomorphism.*
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3. $\mathrm{Pol}(\mathfrak{A})$ *contains a pseudo-Siggers operation:*
   $(\alpha \circ f)(x, y, x, z, y, z) = (\beta \circ f)(y, x, z, x, z, y) : \alpha, \beta \in \overline{\mathrm{Aut}(\mathfrak{A})}$.

$\mathfrak{A}$ is an ω-categorical model-complete core. Then ❶ $\Rightarrow$ ❷ $\Leftrightarrow$ ❸.

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Theorem (Barto, Pinsker '20)

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Remark

❶ $\Leftrightarrow$ ❷ does not hold in general, but it does hold for "reasonable" structures.
(Barto, Kompatscher, Olšák, Pham, Pinsker '17).

Infinite-domain CSP dichotomy

The conjecture

Conjecture (Bodirsky, Pinsker '11)

If $\mathfrak{A}$ is a first-order reduct of a finitely bounded homogeneous structure (FOROFBHS) then CSP($\mathfrak{A}$) is in **P** or it is **NP**-complete,

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Examples: $(\mathbb{Q}; <)$, random graph, random poset, unary ω-categorical structures

Solved for

- reducts of $(\mathbb{N};=)$ (Bodirsky, Kára '08)
- reducts of $(\mathbb{Q};<)$ (Bodirsky, Kára '09)
- reducts of the homogeneous binary branching C-structure (Bodirsky, Jonsson, Pham '16)
- reducts of homogeneous graphs (Bodirsky, Martin, Pinsker, Pongrácz '19)
- reducts of the random poset (Kompatscher, Pham '18)
- reducts of unary ω-categorical structures (Bodirsky, Mottet '18)
- MMSNPs (Bodirsky, Madelaine, Mottet '18)
- reducts of the random tournament (Mottet, Pinsker '21)
- first-order expansions of the homogeneous RCC5 structure (Bodirsky, B. '21)
- hereditarily cellular structures (B. '22)
- first-order expansions of powers of $(\mathbb{Q};<)$ (Bodirsky, Jonsson, Martin, Mottet, Semanišinová '22)
- reducts of random uniform hypergraphs (Mottet, Nagy, Pinsker '23)
- reducts of Johnson graphs (Bodirsky, B. '25)

Interpretation of structures

Definition

$\mathfrak{A}$ interprets $\mathfrak{B}$ if
$\exists I\colon A^d \to B$ surjective partial map such that for all relations R of $\mathfrak{B}$

$$\{(a_1^1, \ldots, a_d^1, \ldots, a_1^k, \ldots, a_d^k) : (I(a_1), \ldots, I(a_k)) \in R\}$$

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Notation

$\mathcal{I}(\mathfrak{A})$: structures interpretable in $\mathfrak{A}$. ($\mathfrak{A} \in \{(\mathbb{N}; =), (\mathbb{Q}; <)\}$).

Johnson graphs

Definition

$\mathcal{J}_k := (\binom{\mathbb{N}}{k}; S_0, S_1, \ldots, S_{k-1})$ where $S_i = \{(a,b) : |a \cap b| = i\}$.

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- Any of the relations S_i defines all the others.
- These are all the primitive oligomorphic actions of $\mathrm{Sym}(\omega)$.

The $k = 1$ case (Equality CSPs)

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Theorem (Bodirsky, Kára '08)

Let $\mathfrak{B}$ be a first-order reduct of $(\mathbb{N}; =)$ (the pure set). Then exactly one of the following holds.

1. *$\mathfrak{B}$ has a 1-element model-complete core, and $\mathrm{CSP}(\mathfrak{B})$ is trivial.*

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2. *$\mathfrak{B}$ is a model-complete core and it has a binary injective polymorphism, and $\mathrm{CSP}(\mathfrak{B}) \in \mathbf{P}$.*

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2. *$\mathfrak{B}$ is a model-complete core and it has a binary injective polymorphism, and* $\mathrm{CSP}(\mathfrak{B}) \in \mathbf{P}$.
3. *$\mathfrak{B}$ is a model-complete core and all polymorphisms are essentially unary, and* $\mathrm{CSP}(\mathfrak{B})$ *is* **NP***-complete.*

Lemma

CSP($\mathcal{J}_k$) *is* **NP***-complete for* $k \geq 2$.

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Theorem (Bodirsky, B. '25)

Let $\mathfrak{B}$ *be a reduct of* $\mathcal{J}_k$*, and let* $\mathfrak{C}$ *be its model-complete core. Then* $\mathfrak{C}$ *is bidefinable with* $\mathcal{J}_\ell$ *for some* $\ell \leq k$*.*

Lemma

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Let $\mathfrak{B}$ *be a reduct of* $\mathcal{J}_k$*, and let* $\mathfrak{C}$ *be its model-complete core. Then* $\mathfrak{C}$ *is bidefinable with* $\mathcal{J}_\ell$ *for some* $\ell \leq k$.

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Johnson graphs

The dichotomy

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I think $1 < \ell < k$ is not possible in the theorem above.

ω-stability

Fact (Bodirsky, B. '25+)

$\mathcal{I}(\mathbb{N};=)$ *and* $\mathcal{I}(\mathbb{Q};<)$ *are NOT closed under taking model-complete cores.*

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Definition (or theorem)

$\mathfrak{A}$ is ω-stable iff $\mathrm{rk}(A)$ exists (not bigger than all ordinals).

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Non-example: $(\mathbb{Q};<)$.

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Fun fact (Cherlin, Lachlan, Harrington '85)

Every ω-categorical ω-stable structure has finite rank.

Lachlan's class

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Strategy for solving the CSP dichotomy:

- Identify primitive structures in $\mathcal{D}$ (Lachlan's class).

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Examples for primitive structure in $\mathcal{D}$:

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We think: this is all and they all have hard CSPs (except for the pure set).

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Remark: this would imply the dichotomy.

Universe of tame ω-categoricity