ALGEBRAIC PROPERTIES OF DYADIC CONVEX POLYGONS

A.B. Romanowska

Warsaw University of Technology,
Warsaw, Poland

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COMMENTS ON THE TITLE

Dyadic rational **numbers** : $m2^{-n}$ for $m, n \in \mathbb{Z}$.

• The ring $\mathbb{D} = \mathbb{Z}[1/2]$ of dyadic numbers is a principal ideal subdomain of the ring $\mathbb{R}$ of real numbers.

• Background:
the theory of idempotent and entropic algebras
- **modes**, in particular
- **affine spaces** over a commutative rings, and
- **convex sets**.

OUTLINE

- Real affine spaces
- Real convex sets and barycentric algebras
- Dyadic affine spaces and dyadic convex sets
- Dyadic intervals
- Dyadic triangles
- Classification of dyadic triangles
- Characterization of dyadic triangles
- Isomorphisms of dyadic triangles

AFFINE SUBSPACES and CONVEX SUBSETS of $\mathbb{R}^n$

$\mathbb{R}$ — the field of reals; $I^\circ :=]0,1[= (0,1) \subset \mathbb{R}$.

The **line** $L_{x,y}$ through $x, y \in \mathbb{R}^n$:
$L_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in \mathbb{R}\}$.

$A \subseteq \mathbb{R}^n$ is a (non-trivial) **affine subspace** of $\mathbb{R}^n$ if together with any two different points x and y it contains the line $L_{x,y}$.

The **line segment** $I_{x,y}$ joining the points x, y:
$I_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in I^\circ\}$.

$C \subseteq \mathbb{R}^n$ is a (non-trivial) **convex subset** of $\mathbb{R}^n$ if together with any two points x and y it contains the line segment $I_{x,y}$.

REAL AFFINE SPACES

Given a (unital) subring R of $\mathbb{R}$.

An **affine space** A **over** R (or **affine** R-**space**) is the algebra

$$\Big(A, \sum_{i=1}^{n} x_i r_i \Big| \sum_{i=1}^{n} r_i = 1\Big).$$

If $2 \in R$ is invertible, this algebra is equivalent to

$$(A, \underline{R}),$$

where

$$\underline{R} = \{\underline{p} \mid p \in R\}$$

and

$$xy\underline{p} = \underline{p}(x, y) = x(1 - p) + yp.$$

THEOREM: The class of affine R-spaces is a variety (equationally defined class of algebras).

REAL CONVEX SETS and BARYCENTRIC ALGEBRAS

Let R be a subfield of $\mathbb{R}$ and $I^o :=]0, 1[= (0, 1) \subset R$.

Convex subsets of affine R-spaces $(A, \underline{R})$ are I^o-subreducts $(A, \underline{I}^o)$ of $(A, \underline{R})$

Real (convex) **polytopes** are finitely generated convex sets; real **polygons** are finitely generated convex subsets of R^2.

The class C of convex sets generates the variety BA of **barycentric algebras**.

DYADIC CONVEX SETS

Consider the ring

$$\mathbb{D} = \mathbb{Z}[1/2] = \{m2^{-n} \mid m, n \in \mathbb{Z}\}$$

of dyadic rational numbers.

A **dyadic convex set** is the intersection of a real convex set in $\mathbb{R}^k$ with the space $\mathbb{D}^k$.

A **dyadic polytope** is the intersection of the space $\mathbb{D}^k$ with a real polytope in $\mathbb{R}^k$ whose vertices lie in $\mathbb{D}^k$.

A **dyadic triangle** and **dyadic polygon** are (respectively) the intersection with $\mathbb{D}^2$ of a triangle or polygon in $\mathbb{R}^2$ having vertices in $\mathbb{D}^2$.

Dyadic intervals form the one-dimensional analogue.

REAL VERSUS DYADIC

• Real polytopes are barycentric algebras $(A, \underline{I}^o)$.

• Dyadic polytopes are algebras $(A, \underline{\mathbb{D}}_1^o)$,
where $\mathbb{D}_1^o =]0, 1[\cap \mathbb{D}$.

Proposition [Ježek, Kepka, 1976]: Each dyadic polytope $(A, \underline{\mathbb{D}}_1^o)$ is equivalent to
$(A, \cdot) = (A, \frac{1}{2}(x + y))$.

Note that the operation $\cdot$ is:

idempotent: $x \cdot x = x$;
commutative: $x \cdot y = y \cdot x$;
entropic (medial): $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$.

Hence: dyadic polytopes are
commutative binary modes (or CB-modes).

REAL VERSUS DYADIC, cont.

• All real intervals are isomorphic (to the interval $I = S_1$). Each is generated by its ends.
• All real triangles are isomorphic (to the simplex S_2). Each is generated by its vertices.

NOT TRUE for dyadic intervals and dyadic triangles.

Example: The dyadic interval $[0, 3]$ is generated by no less than 3 elements. The minimal set of generators is given e.g. by the numbers $0, 2, 3$.

• The class of convex subsets of affine $\mathbb{R}$-spaces is characterized as the subquasivariety of cancellative barycentric algebras.

NOT TRUE for the class of convex dyadic subsets of affine $\mathbb{D}$-spaces.
(K. Matczak, R., 2004)

SOME PROBLEMS

Which characteristic properties of real polytopes (in particular polygons) carry over to dyadic polytopes (polygons)?

Note that dyadic polygons are described using dyadic intervals and dyadic triangles.

Problem: Classify all dyadic intervals and all dyadic triangles up to isomorphism.

Isomorphisms of dyadic polytopes are described as restrictions of automorphisms of the affine dyadic spaces, members of the affine group $GA(n, \mathbb{D})$.

Problems:
Are all dyadic intervals finitely generated?
Are all dyadic triangles finitely generated?

DYADIC INTERVALS

An initial classification of dyadic intervals was given by K. Matczak, R., J.D.H. Smith in 2011.

THEOREM: Each interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$ (is **of type** k), where k is an odd positive integer. Two such intervals are isomorphic precisely when their right hand ends are equal.

The interval $[0, 1]$ is generated by its ends. For each positive integer k, and each integer r, the intervals $[0, k]$ and $[0, k2^r]$ are isomorphic.

THEOREM: Each dyadic interval of type $k > 1$ is minimally generated by three elements.

GENERATION OF DYADIC TRIANGLES

The first classification of dyadic triangles, by K. Matczak, R., J.D.H. Smith in 2011, implied the following.

THEOREM: Each dyadic triangle is finitely generated.

COROLLARY: Each dyadic polygon is finitely generated.

GENERATION OF DYADIC POLYTOPES

Later results:

THEOREM [Matczak, Mućka, R., 2023]: Each dyadic polytope is finitely generated.

THEOREM [Matczak, Mućka, R., 2023]: A subgroupoid D of $(\mathbb{D}^n, \circ)$ is finitely generated precisely when its dyadic convex hull is a polytope with the same vertices and the same interior as D.

DYADIC TRIANGLES AND THEIR BOUNDARY TYPES

The types m, n, k of sides of a triangle determine its **boundary type** (m, n, k).

Proposition: The triangles of **right type** (i.e., with shorter sides parallel to the coordinate axes) are determined uniquely up to isomorphism by their boundary type.

The boundary type does not determine a general dyadic triangle.

Example: There are infinitely many pairwise non-isomorphic triangles of boundary type $(1, 1, 1)$.

There are triangles in $\mathbb{D}^2$ not isomorphic to right triangles.

TYPES OF DYADIC TRIANGLES

Automorphisms of the dyadic plane $\mathbb{D}^2$ are described as elements of the affine group $GA(2, \mathbb{D})$. These automorphisms transform a triangle in the plane $\mathbb{D}^2$ onto an isomorphic triangle.

Lemma: Each dyadic triangle is isomorphic to a (**pointed**) triangle ABC contained in the first quadrant, with one vertex, say A, located at the origin, and the vertices B and C having non-negative integer coordinates.

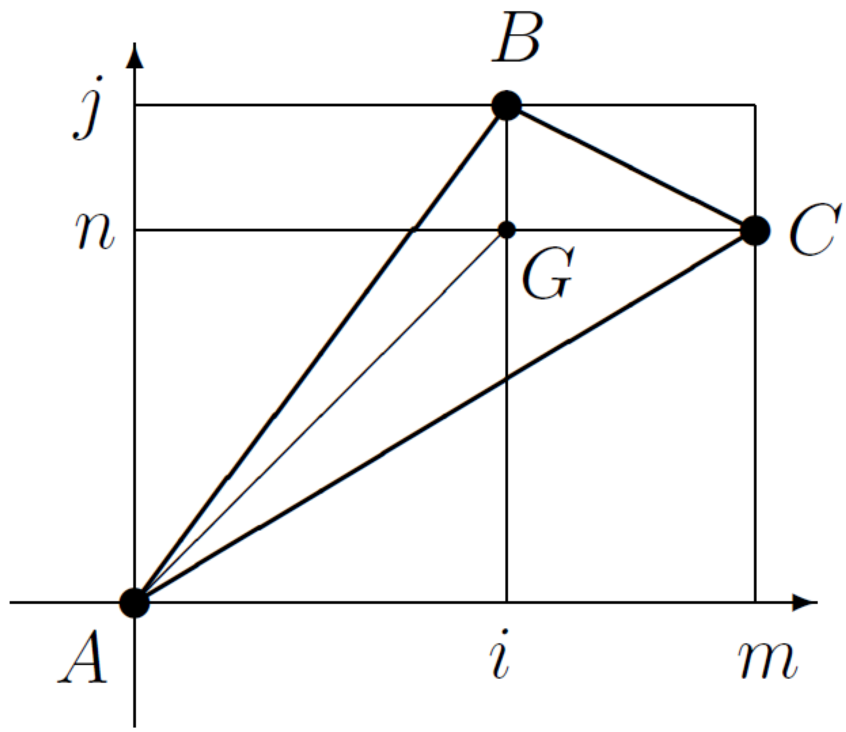
TYPES OF DYADIC TRIANGLES, cont.

If $B = (i, j)$ and $C = (m, n)$, then i, j, m may be chosen so that $0 \leq i < m$ and $0 \leq n < j$.

Such a triangle is denoted $T_{i,j,m,n}$.

A triangle $T_{0,j,m,0}$ is a right triangle.

A triangle $T_{i,j,m,0}$ is a **hat** triangle.
(One of its sides is parallel to a coordinate axis).


B
j
n
C
G
A
i
m

OLDER CLASSIFICATION

When a vertex A of a dyadic triangle located at the origin is chosen, the triangle ABC comes in one of three types.

THEOREM [Matczak, Mućka, R., Smith, 2011, 2019, 2023]: Each pointed dyadic triangle ABC comes as a triangle $T_{i,j,m,n}$ in one of the following three types:

(a) **right triangles** $T_{0,j,m,0}$
(j and m odd and $j \leq m$);

(b) **hat triangles** $T_{i,j,m,0}$
($0 < i \leq m/2$, odd $j > 1$, and $\gcd\{i, j\} \neq j$);

(c) **other triangles** $T_{i,j,m,n}$
(neither of i, n is zero, moreover $j \leq m$, $\gcd\{i, j\} \notin \{i, j, 1\}$ and $\gcd\{m, n\} \notin \{m, n, 1\}$).

FURTHER PROPERTIES

Proposition: Let $\iota : T_1 \to T_2$ be an isomorphism between dyadic triangles.
For $i = 1, 2$, let P_i be the area of the convex $\mathbb{R}$-hull of T_i.
Then $P_1/P_2 = 2^k$ for some $k \in \mathbb{Z}$.

Lemma: Let $P \in \mathbb{D}^2$ with integral coordinates and not on any axes. There is a $\mathbb{D}$-module automorphism taking P onto a point of one of the axes, again with integral coordinates.

NEW CLASSIFICATION

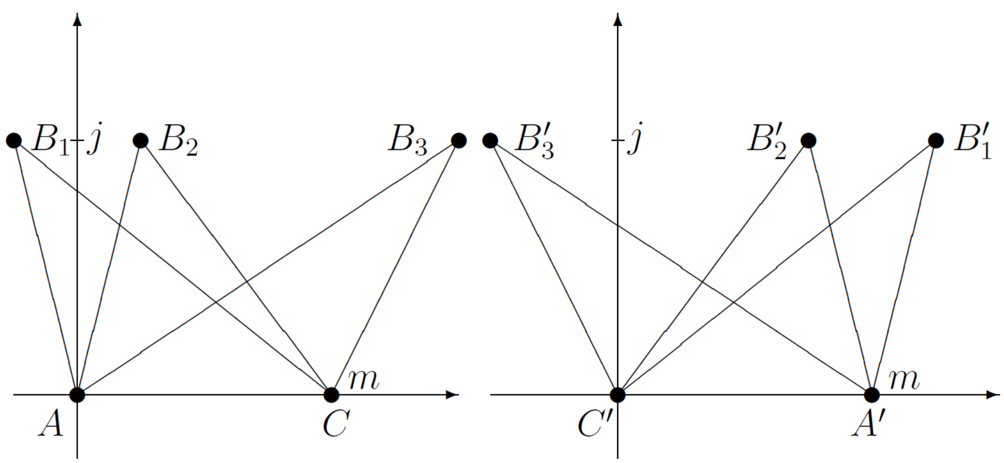
The following results were obtained recently by A. Mućka and R.

Proposition: Let T be a dyadic triangle in the dyadic plane $\mathbb{D}^2$. Then T is isomorphic to each of the following triangles with vertices having integral coordinates:

(a) A triangle ABC contained in the upper half-plane of $\mathbb{D}^2$ with one vertex, say A, located at the origin, and another vertex, say $C = (m, 0)$, on the positive part of the x-axis;

(b) A triangle ABC contained in the first quadrant of $\mathbb{D}^2$ with A and C defined as above, and the third vertex $B = (i, j)$ such that $0 \leq i \leq m$.

From now on, $T_{i,j,m,0}$ denotes a triangle with positive integers j and m, and any integer i.



B_1
j
B_2
B_3
m
A
C
B'_3
j
B'_2
B'_1
m
C'
A'

REPRESENTATIVE HATS

Proposition: Each triangle $T_{i,j,m,0}$ is isomorphic to a triangle $T_{i',j',m',0}$, where all i', j', m' are odd.

A triangle $T_{i,j,m,0}$, with all i, j, m odd, is denoted by $T_{i,j,m}$ and called a **representative hat**.

THEOREM: Each dyadic triangle in the dyadic space $\mathbb{D}^2$ is isomorphic to a representative hat.

Note: Each dyadic triangle is isomorphic to three representative hats with the same vertex ordering, and three isomorphic hats with the reverse ordering. They correspond to the six permutations of the vertices.

POINTED ISOMORPHISMS OF REPRESENTATIVE HATS

Isomorphism of representative hats is considered as a **pointed oriented isomorphism**, i.e. an isomorphism preserving the pointed vertex and the vertex orientation.

Proposition: Consider two representative hats $T = ABC$ and $T' = A'B'C'$ with vertices $A = A' = (0, 0)$, $B = (i, j), C = (m, 0)$ and $B' = (i', j'), C' = (m', 0)$. Let $\iota : T \to T'$ be a mapping taking B to B' and C to C'.

Then $\iota : T \to T'$ is a pointed oriented isomorphism precisely when $j' = j$, $m' = m$ and $i' = i + jk$ for some integer k.

CHARACTERIZATION OF POINTED DYADIC TRIANGLES

Corollary: For any positive odd integers j and m, there are j (pointed) isomorphism classes of representative hats $T_{i,j,m}$. Each class is represented by a unique $T_{i,j,m}$, where $i \in \{1, 3, \ldots, 2j-1\}$.

Corollary: Each pointed dyadic triangle is (pointed oriented) isomorphic to a unique $T_{i,j,m}$, where j and m are positive odd integers and $i \in \{1, 3, \ldots, 2j-1\}$.

A triple (i, j, m) of integers, where j and m are positive and odd and $i \in \{1, 3, \ldots, 2j-1\}$, is called an **encoding triple**.

THEOREM: Two pointed dyadic triangles with the same orientation of vertices are isomorphic if and only if they have the same encoding triples.

ISOMORPHISMS OF REPRESENTATIVE HATS

Necessary conditions

Proposition: If two representative hats $T = T_{i,j,m}$ and $T' = T_{i',j',m'}$ are isomorphic, then the following two conditions hold

(a) The hats T and T' have equal or oppositely oriented boundary types.
(b) The areas of the convex $\mathbb{R}$-hulls $\mathrm{conv}_{\mathbb{R}}(T)$ of T and $\mathrm{conv}_{\mathbb{R}}(T')$ of T' are equal.

Moreover, if a mapping $\iota : T \to T'$ is an isomorphism, then it maps the vertex set $V(T)$ of T onto the vertex set $V(T')$ of T'.

None of the necessary conditions of this Proposition is sufficient.

Sufficient conditions

Each representative hat is isomorphic to an **almost representative hat** $T_{i,j,m}$, i.e. with an arbitrary integer i, and both j and m positive and odd integers.

THEOREM Let T and T' be two almost representative hats, the first one a hat $T_{i,j,m}$ with vertices A, B, C, and the second a hat $T_{k,l,n}$ with vertices A', B', C'.
Let $\iota : T \to T'$ be a mapping taking the triple (A, B, C) to the triple (X, Y, Z), where (X, Y, Z) is a permutation of (A', B', C') .

Assume that both hats have equal or oppositely oriented boundary types, and equal areas of their convex $\mathbb{R}$-hulls.

Then ι is an isomorphism precisely when one of six elementary number-theoretical conditions on i, j, m, k, l, n holds, corresponding to the six permutations (X, Y, Z) of A', B', C'.

THANK YOU !

Some references

G. Czédli, A. Romanowska, Generalized convexity and closure conditions, *Int. J. of Algebra and Computation* **23** (2013) 1805–1835.

J. Ježek and T. Kepka, *Medial Groupoids,* Academia, Praha, 1983.

K. Matczak, A. Mućka, A. Romanowska, Finitely generated dyadic convex sets, *Int. J. of Algebra and Computation*, **33** (2023) 585–615.

K. Matczak, A. Romanowska, Quasivarieties of cancellative commutative binary modes, *Studia Logica* **78** (2004) 321–335.

K. Matczak, A. Romanowska, J. D. H. Smith, Dyadic polygons, *Int. J. of Algebra and Computation* **21** (2011) 387–408.

A. Mućka, A. Romanowska, Geometry of dyadic polygons I: The structure of dyadic triangles, preprint, 2025.

A. Mućka, A. Romanowska, Geometry of dyadic polygons II: Isomorphisms of dyadic triangles, preprint, 2025.

W. D. Neumann, On the quasivariety of convex subsets of affine spaces, *Arch. Math. (Basel)* **21** (1970) 11–16.

A. B. Romanowska, J. D. H. Smith, *Modal Theory,* Heldermann Verlag, Berlin, 1985.

A. B. Romanowska, J. D. H. Smith, *Modes,* World Scientific, Singapore, 2002.