(Right-)preordered groups from a categorical perspective

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Joint work with Maria Manuel Clementino and Nelson Martins-Ferreira

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A right-preordered group is a group G equipped with a preorder $\leq$ which is compatible with the group operation only on the right:

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We denote by OrdGrp (resp. by ROrdGrp) the category of preordered (resp. right-preordered) groups and monotone homomorphisms.

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$$P\colon \mathsf{OrdGrp} \to \mathsf{Mon}_{\mathrm{can}}, \quad P_R\colon \mathsf{ROrdGrp} \to \mathsf{Mon}_{\mathrm{can}}$$

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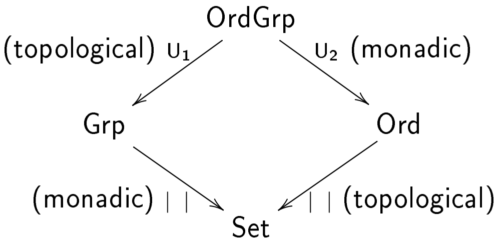
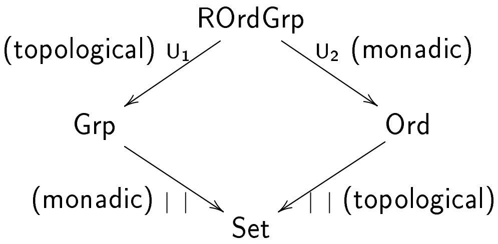
Forgetful functors





Forgetful functors

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, an F-structured source is an object $D \in \mathcal{D}$ together with a family of arrows $(f_i\colon D \to F(C_i))_{i \in I}$. The functor F is topological if every F-structured source $(D, (f_i)_{i \in I})$ has a unique F-initial lifting, i.e. a unique object $C \in \mathcal{C}$, with arrows $g_i\colon C \to C_i$, such that $F(C) = D$ and $F(g_i) = f_i$.

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U_1 from OrdGrp is a topological functor: let $(f_i\colon G \to X_i)_{i\in I}$ be a family of group homomorphisms, with X_i, $i \in I$, preordered groups. Then $P_G = \{y \in G \mid f_i(y) \in P_{X_i} \text{ for every } i \in I\}$ is a submonoid of G closed under conjugation, and this defines the U_1-initial lifting for (f_i).

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$U_2 \colon \mathrm{OrdGrp} \to \mathrm{Ord}$ has a left adjoint. It sends a preorder A to the free group $\mathrm{F}(A)$ on the set A, equipped with the preorder determined by the submonoid of $\mathrm{F}(A)$ obtained by closing under addition and conjugation the set of the elements of the form $b - a$ for all $a, b \in A$ such that $a \leq b$.

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Coequalizers: given a pair of morphisms $f, g \colon X \to Y$, let $q \colon \mathsf{U}_1(Y) \to Q$ be the coequalizer in Grp of $\mathsf{U}_1(f), \mathsf{U}_1(g)$. Putting $P_Q = q(P_Y)$, we get that (Q, P_Q) a preordered group, and $q \colon Y \to (Q, P_Q)$ is a morphism in OrdGrp.

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Coproducts: given two preordered groups X and Y, its coproduct is the free product of $U_1(X)$ and $U_1(Y)$ in Grp equipped with the positive cone obtained as the closure, under addition and conjugation, of the disjoint union of P_X and P_Y.

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The situation for ROrdGrp is similar.

Monomorphisms and epimorphisms

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4. f is a regular monomorphism (i.e. an equalizer of a parallel pair of morphisms) if and only if f is injective and $P_X = f^{-1}(P_Y)$.

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An object P in a category $\mathcal{C}$ is regular projective if the functor $\mathcal{C}(P,-)$ preserves regular epimorphisms. P is a regular generator if, for any set S, the S-indexed copower $S \cdot P$ of P exists and, for any object X of $\mathcal{C}$, the canonical morphism $\mathcal{C}(P,X) \cdot P \to X$ is a regular epimorphism.

Theorem (Pedicchio-Vitale)

A category is a quasivariety if and only if it is regular, it has a regular projective, regular generator, and it has coequalizers of equivalence relations.

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Both OrdGrp *and* ROrdGrp *are quasivarieties.*

Proof: the coproduct of $(\mathbb{Z}, 0)$ and $(\mathbb{Z}, \mathbb{N})$ is a regular projective, regular generator.

Unital categories

Definition (Bourn)

A pointed, finitely complete category is unital if, for every pair of objects X, Y*, the canonical morphisms* $\langle 1, 0\rangle\colon X \to X \times Y$ *and* $\langle 0, 1\rangle\colon Y \to X \times Y$ *are jointly extremally epimorphic.*

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This means that, for every commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,0\rangle} & X \times Y & \xleftarrow{\langle 0,1\rangle} & Y, \\ & {\scriptstyle f}\searrow & \uparrow{\scriptstyle m} & \swarrow{\scriptstyle g} & \\ & & Z & & \end{array}$$

where m is a monomorphism, m is an isomorphism.

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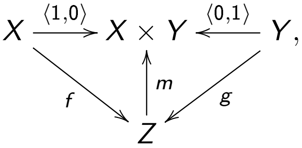

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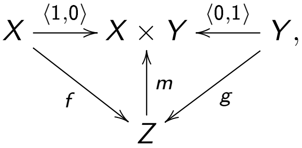

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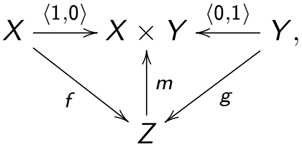

where m is a monomorphism, m is a group isomorphism. Its inverse is given by $t(x, y) = f(x) + g(y)$, and it is monotone.

Jointly extremally epimorphic families

In ROrdGrp a family of morphisms $(f_i \colon X_i \to X)_{i \in I}$ is jointly extremally epimorphic if and only if

1. $(f_i \colon X_i \to X)_{i \in I}$ is jointly extremally epimorphic in Grp;
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Commutative and abelian objects

An object A in a unital category is commutative if there exists a morphism $\varphi\colon A \times A \to A$ making the following diagram commutative:

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A & \xrightarrow{\langle 1,0 \rangle} & A \times A & \xleftarrow{\langle 0,1 \rangle} & A \\
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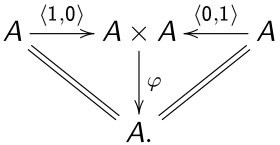

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The full subcategory of abelian objects in both OrdGrp *and* ROrdGrp *is the category of monomorphisms of abelian groups.*

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we obtain a morphism of split extensions in OrdGrp and in ROrdGrp:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\langle 1,0\rangle} & \mathbb{Z} \times_p \mathbb{Z} & \underset{\pi_2}{\overset{\langle 0,1\rangle}{\leftrightarrows}} & \mathbb{Z} \\ \| & & \downarrow{\scriptstyle 1_{\mathbb{Z}\times\mathbb{Z}}} & & \| \\ \mathbb{Z} & \xrightarrow{\langle 1,0\rangle} & \mathbb{Z} \times_l \mathbb{Z} & \underset{\pi_2}{\overset{\langle 0,1\rangle}{\leftrightarrows}} & \mathbb{Z} \end{array}$$

whose middle component is not an isomorphism, because its inverse is not monotone.

Definition

A point (i.e. a split epimorphism with a fixed section) $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *with kernel* $k \colon X \to A$ *in a pointed finitely complete category* $\mathcal{C}$ *is strong if* k *and* s *are jointly extremally epimorphic.*

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This happens if and only if the split short five lemma holds in $\mathcal{C}$.

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$\mathcal{C}$ is a Mal'tsev category if and only if, for every pullback of split epimorphisms as in the following diagram

$$\begin{array}{ccc} A \times_Y C & \underset{\pi_C}{\overset{\langle sg, 1_C \rangle}{\leftrightarrows}} & C \\ {\scriptstyle \pi_A} \downarrow \uparrow {\scriptstyle \langle 1_A, tf \rangle} & & {\scriptstyle g} \downarrow \uparrow {\scriptstyle t} \\ A & \underset{f}{\overset{s}{\leftrightarrows}} & Y, \end{array}$$

the morphisms $\langle 1_A, tf \rangle$ and $\langle sg, 1_C \rangle$ are jointly extremally epimorphic.

Mal'tsev and protomodular objects

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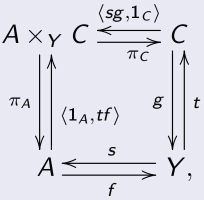

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Proposition

Every cancellative, strongly unital monoid is a group.

Mal'tsev and protomodular objects

Theorem

Both in OrdGrp *and in* ROrdGrp, *the following conditions are equivalent:*

1. *Y is a protomodular object;*
2. *Y is a Mal'tsev object;*
3. *Y is a strongly unital object;*
4. *P_Y is a group;*
5. *the preorder relation on Y is an equivalence relation.*