Set-theoretic solutions of the Yang-Baxter and their reflections

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This talk will provide basic overviews on

- the *s*et-theoretic Yang-Baxter equation with a focus on self-distributive structures;
- the associated *s*et-theoretic reflection equation, with final remarks on some results obtained in:

A. Albano, M. Mazzotta, P. Stefanelli, *Reflections to set-theoretic solutions of the Yang-Baxter equation*, J. Algebra 676, 106-138 (2025).

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The Yang–Baxter equation

A solution to the Yang-Baxter equation is a pair (V, R) with

- V a vector space (over a field Λ);
- $R \in \mathrm{End}_\Lambda(V \otimes V)$ satisfying the following identity:

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)$$

C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. 19 (1967), 1312–1315.

R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Physics 70 (1972), 193–228.

The set-theoretic Yang–Baxter equation

A solution to the *set-theoretic Yang–Baxter equation* is a pair (D, r) with

- D a (non-empty) set;
- $r : D \times D \to D \times D$ satisfying the following identity:

$$(r \times \mathrm{id}_D)(\mathrm{id}_D \times r)(r \times \mathrm{id}_D) = (\mathrm{id}_D \times r)(r \times \mathrm{id}_D)(\mathrm{id}_D \times r)$$

V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, in Quantum groups, (Springer) Lecture Notes in Math. 1510 (1990), 1-8.

Notation

From now on, the term *solution* will always refer to the *set-theoretic* YBE.

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If we write $r(a, b) = (\lambda_a(b), \rho_b(a))$ where $\lambda_a, \rho_b : D \to D$ are maps, then

- r is *left non-degenerate* if λ_a is bijective, for all $a \in D$;
- r is *right non-degenerate* if ρ_b is bijective, for all $b \in D$;
- r is *non-degenerate* if both left and right non-degenerate.

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(D, r) is a solution if and only if the following hold for all $x, y, z \in D$:

$$\lambda_x \lambda_y(z) = \lambda_{\lambda_x(y)} \lambda_{\rho_y(x)}(z) \tag{YB1}$$

$$\rho_{\lambda_{\rho_y(x)}(z)} \lambda_x(y) = \lambda_{\rho_{\lambda_y(z)}(x)} \rho_z(y) \tag{YB2}$$

$$\rho_x \rho_y(z) = \rho_{\rho_x(y)} \rho_{\lambda_y(x)}(z) \tag{YB3}$$

Examples

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determines the **trivial solution** on D. It is bijective non-degenerate.
- If $f, g : D \to D$ are *commuting* maps then
$$r(x, y) = (f(y), g(x))\,,$$
determines a **Lyubashenko solution**. It is left non-degenerate (resp. right non-degenerate) if and only if f (resp. g) is bijective.

Self-distributive structures

Let $(D, \triangleright)$ be a magma and consider $r(x, y) = (y, y \triangleright x)$, for all $x, y \in D$.

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- $(D, \triangleright)$ is a rack if $L_x : D \ni y \mapsto x \triangleright y \in D$ is bijective, for all $x \in D$;
- $(D, \triangleright)$ is a quandle if it is a rack such that $x \triangleright x = x$, for all $x \in D$.

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D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), 37-65.

S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119 (1982), 78-88.

Examples

- (Conjugation quandle) Let G be a group and define Conj(G) by

$$x \triangleright y = x^{-1} y x \,, \quad x, y \in G \,.$$

- (Core quandle) Let G be a group and define Core(G) by

$$x \triangleright y = x y^{-1} x \,, \quad x, y \in G \,.$$

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- (Core quandle) Let G be a group and define $\mathrm{Core}(G)$ by
$$x \triangleright y = xy^{-1}x\,, \quad x, y \in G\,.$$
- Let G be a group, fix $a \in G$ and define a binary operation by setting
$$x \triangleright y = yx^{-1}ax\,, \quad x, y \in G\,.$$
Then, $(G, \triangleright)$ is a rack without idempotents unless $a \in Z(G)$.

If $(D, \triangleright)$ is a shelf, then the solution:

$$r : D \times D \to D \times D\,,\; r(x, y) = (y, y \triangleright x).$$

is known to be of *left derived* type.

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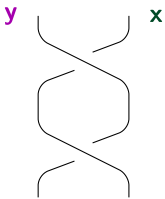
Can we mimic this behaviour for a general solution (D, r)?

Derived solutions

If (D, r) is a left non-degenerate solution, its square r^2 can be represented as follows:



A. V. Soloviev, *Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation*, Math. Res. Lett. 7 (2000), 577–596.

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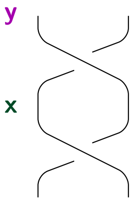

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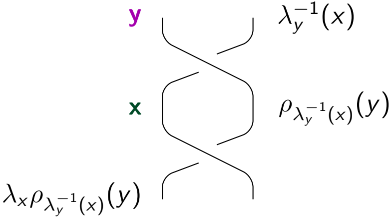

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Let (D, r) be a left non-degenerate solution and define

$$x \triangleright_r y = \lambda_x \rho_{\lambda_y^{-1}(x)}(y)\,, \quad x, y \in D\,.$$

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- For all $a \in D$ we have $\lambda_a \in \mathrm{Aut}(D, \triangleright_r)$.

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The solution (D, r') defined by $r'(x, y) = (y, y \triangleright_r x)$ is called the *left derived solution* associated with (D, r).

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[Doikou, Rybołowicz, Stefanelli (2024)]

If we define $F : D \times D \to D \times D\,, F(x, y) = (x, \lambda_x(y))$, then

$$Fr = r'F\,.$$

We say that (D, r) and (D, r') are *Drinfel'd isomorphic*.

What if we are interested in a stronger notion of isomorphism?

Let (D, r), (E, s) be solutions. A map $f : D \to E$ is a *st*-morphism if

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There is a category **Sol** whose

- *objects* are all solutions;
- *morphisms* are all st-morphisms between solutions.

Non-degenerate solutions form a full subcategory $\mathbf{Sol_{ND}}$.

Skew braces

A *(left) skew brace* is a set D equipped with two group operations $+$, $\circ$ satisfying the following identity, for all $x, y, z \in D$:

$$x \circ (y + z) = x \circ y - x + x \circ z\,.$$

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Notable examples:

- Radical rings: $(R, +, \circ)$ where $x \circ y = x + xy + y$, for all $x, y \in R$;
- Regular subgroups of Hol(G,+).

W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307 (2007), 153–170.

L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comp. 86 (2017), 2519–2534.

Skew braces and solutions

If $(D, +, \circ)$ is a skew brace, then there exists a bijective non-degenerate solution (D, r_D) defined by setting, for all $x, y \in D$:

$$r_D(x, y) = \left(-x + x \circ y, (-x + x \circ y)^{-} \circ x \circ y\right).$$

F. Cedó, E. Jespers, J. Okniński, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. 327 (2014), 101–116.

F. Catino, I. Colazzo, P. Stefanelli, *Semi-braces and the Yang-Baxter equation*, J. Algebra 483 (2017), 163-187.

D. Bachiller, *Solutions of the Yang-Baxter equation associated to skew left braces, with applications to racks*, J. Knot Theory Ramifications 27 (2018).

A. Smoktunowicz, L. Vendramin, *On skew braces (with an appendix by N. Byott and L. Vendramin)*, J. Comb. Algebra 2 (2018), pp. 47–86.

Let (D, r) be a non-degenerate solution and let $r(x, y) = (\lambda_x(y), \rho_y(x))$. Its *structure group* is defined as

$$G(D, r) = \langle\, x \in D \mid xy = \lambda_x(y)\rho_y(x) \,\rangle$$

$$\mathbf{Sol_{ND}} \ni (D, r) \longrightarrow G(D, r) \in \mathbf{Skb}$$

$$\perp$$

$$\mathbf{Sol_{ND}} \ni (B, r_B) \longleftarrow (B, +, \circ) \in \mathbf{Skb}$$

T. Gateva-Ivanova, M. Van den Bergh, *Semigroups of I-type*, J. Algebra 206 (1998), 97–112.

P. I. Etingof, T. Schedler and A. V. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. 100 (1999), no. 2, 169–209.

J.-H. Lu, M. Yan and Y. Zhu, On the set-theoretical Yang-Baxter equation, Duke Math. J. 104 (2000), 1–18.

A. Soloviev, *Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation*, Math. Res. Lett. 7 (2000), 577-596.

A general framework

Let $(D,\triangleright)$ be a shelf. A map $\lambda: D \to \mathrm{Aut}(D,\triangleright)$ is a *twist* of $(D,\triangleright)$ if

$$\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\lambda^{-1}_{\lambda_x(y)}(\lambda_x(y)\triangleright x)},$$

holds, for all $x, y \in D$.

A. Doikou, B. Rybołowicz, P. Stefanelli, *Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal R-matrices*, J. Phys. A, Math. Theor. 57 (2024).

[Doikou, Ribołowicz, Stefanelli (2024)]

Let $(D, \triangleright)$ be a shelf and $\lambda : D \to \mathrm{Aut}(D, \triangleright)$ a map. Then

$$r : D \times D \to D \times D\,,\ r(x, y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(\lambda_x(y) \triangleright x))$$

is a left non-degenerate solution if and only λ is a twist.

Whenever λ is a twist we have that $\triangleright_r = \triangleright$. It follows that all *left non-degenerate solutions* can be obtained in this way.

For example, if $(D, +, \circ)$ is a skew brace and (D, r) its associated solution, then $(D, \triangleright_r) = \mathrm{Conj}(D, +)$.

What happens when a dynamical system admits a boundary that interacts with the particles in motion?

The Reflection Equation

Let (D, r) be a solution. A map $\kappa : D \to D$ is a *solution to the set-theoretic reflection equation*, or simply a *reflection*, for (D, r) if the following holds:

$$r\ (\mathrm{id}_D \times \kappa)\ r\ (\mathrm{id}_D \times \kappa) = (\mathrm{id}_D \times \kappa)\ r\ (\mathrm{id}_D \times \kappa)\ r$$

I. Cherednik, *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. 61 (1984), no. 1, 35-44.

E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A 21 (1988), no. 10, 2375-2389.

V. Caudrelier, Q. C. Zhang, *Yang-Baxter and reflection maps from vector solitons with a boundary*, Nonlinearity 27 (2014), no. 6, 1081-1103.

Notation:

$\mathcal{K}(D, r)$ is the set of all reflections for (D, r).

$\mathcal{K}_{\mathsf{bij}}(D, r)$ is the set of all bijective reflections for (D, r).

Example

Let $f, g : D \to D$ be commuting maps, i.e. $fg = gf$ and consider the associated Lyubashenko solution (D, r):

$$r(x, y) = (f(y), g(x)), \quad x, y \in D.$$

A map $\kappa : D \to D$ satisfies $\kappa \in \mathcal{K}(D, r) \iff \kappa(fg) = (fg)\kappa$.

Different approaches

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- [Smoktunowicz, Vendramin, Weston (2020)] employs general methods in the language of *braces* to produce families of reflections.
- [Doikou, Smoktunowicz (2021)] investigates connections between set-theoretic YBE, RE and quantum integrable systems.

- [Lebed, Vendramin (2022)] introduces the notion of κ-*derived solution* and focuses on reflections for involutive non-deg. solutions.

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- [Ferri (2025x)] introduces the notion of *group reflection* for a braided group and studies the extension of reflections to the structure group.

λ-centralizing and ρ-invariant maps

Let D be a set and consider a map $\sigma : D \to D^D\,, \sigma(a) = \sigma_a$.
A map $\kappa : D \to D$ is called

- σ-centralizing if $\sigma_a \kappa = \kappa \sigma_a$, for all $a \in D$;
- σ-invariant if $\sigma_{\kappa(a)} = \sigma_a$, for all $a \in D$.

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Let (D, r) be a solution and write $r(a, b) = (\lambda_a(b), \rho_b(a))$. Then:

- [Smoktunowicz, Vendramin, Weston (2020)] if r is involutive left non-degenerate then

$$\kappa : D \to D \text{ is } \lambda\text{-centralizing} \implies \kappa \in \mathcal{K}(D, r);$$

- [Lebed, Vendramin (2022)] if r is involutive right non-degenerate then

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Let (D, r) be left non-degenerate and $\kappa : D \to D$ be λ-centralizing. Then

$$\kappa \in \mathcal{K}(D, r) \iff \begin{cases} \forall\, x, y \in D \;\; \kappa L_{L_x(y)}(x) = L_{\kappa L_x(y)}\kappa(x) \\ \forall\, x \in D \;\; \kappa L_x = \kappa L_{\kappa(x)} \end{cases} \iff \kappa \in \mathcal{K}(D, r_\triangleright)$$

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If (D, r) is bijective non-degenerate and $\kappa : D \to D$ is λ-centralizing, then

$$\kappa \in \mathcal{K}(D, r) \iff \begin{cases} \kappa \in \mathrm{End}\,(D, \triangleright_r) \\ \forall\, x \in D \quad L_{\kappa(x)} \kappa = \kappa L_{\kappa(x)} \end{cases}$$

Reflections for left non-degenerate solutions

Recall that for all $x, y \in D$ we have $x \triangleright_r y = L_x(y) = \lambda_x \rho_{\lambda_y^{-1}(x)}(y)$.

[AA, Mazzotta, Stefanelli (2025)]

Let (D, r) be left non-degenerate and $\kappa : D \to D$ be λ-centralizing. Then

$$\kappa \in \mathcal{K}(D, r) \iff \begin{cases} \forall\, x, y \in D \;\; \kappa L_{L_x(y)}(x) = L_{\kappa L_x(y)}\kappa(x) \\ \forall\, x \in D \;\; \kappa L_x = \kappa L_{\kappa(x)} \end{cases} \iff \kappa \in \mathcal{K}(D, r_\triangleright)$$

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If (X, r) is *involutive* left non-degenerate then $(D, \triangleright_r)$ is *trivial*.

Reflections for right non-degenerate solutions

If (D, r) is a right non-degenerate solution, one can consider the right rack $(D, \triangleleft_r)$ where $a \triangleleft_r b := \rho_a \lambda_{\rho_b^{-1}(a)}(b)$, for all $a, b \in D$.

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Let (D, r) be right non-degenerate and $\kappa : D \to D$ be a ρ-invariant map.

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Reflections for solutions of derived type

Let $(D, \triangleright)$ be a rack. The *left multiplication group* of $(D, \triangleright)$ is the normal subgroup of $\operatorname{Aut}(D, \triangleright)$ defined by

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If $(D, \triangleright)$ is a rack, then

$$\mathcal{K}(D, r_\triangleright) \supseteq C_{\mathrm{End}(D,\triangleright)}\left(\mathrm{LMlt}\left(D, \triangleright\right)\right) \, ,$$
$$\mathcal{K}_{\mathrm{bij}}(D, r_\triangleright) = C_{\mathrm{Aut}(D,\triangleright)}\left(\mathrm{LMlt}\left(D, r_\triangleright\right)\right) \, .$$

Related and open problems

- For a class of racks $\mathcal{F}$ and $X \in \mathcal{F}$, determine $\mathrm{Aut}(X)$ and $\mathrm{LMlt}(X)$.

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  Lately, in [Bonatto, Spaggiari (2025)] detailed investigation on $\mathrm{LMlt}(\mathrm{Core}(G))$ and on important properties of core quandles.
- Are there broader classes of maps beyond the λ-centralizing or ρ-invariant ones where to look for reflections?

Thank you for your attention!

Grazie per l'attenzione!

References

- A. Albano, M. Mazzotta, P. Stefanelli, *Reflections to set-theoretic solutions of the Yang-Baxter equation*, J. Algebra 676, 106-138 (2025).
- D. Bachiller, *Solutions of the Yang-Baxter equation associated to skew left braces, with applications to racks*, J. Knot Theory Ramifications 27 (2018).
- R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Physics 70 (1972), 193–228.
- A. Smoktunowicz, L. Vendramin, *On skew braces (with an appendix by N. Byott and L. Vendramin)*, J. Comb. Algebra 2 (2018), pp. 47–86.
- F. Catino, I. Colazzo, P. Stefanelli, *Semi-braces and the Yang-Baxter equation*, J. Algebra 483 (2017), 163-187.
- V. Caudrelier, Q. C. Zhang, *Yang-Baxter and reflection maps from vector solitons with a boundary*, Nonlinearity 27 (2014), no. 6, 1081-1103.
- F. Cedó, E. Jespers, J. Okniński, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. 327 (2014), 101–116.
- I. Cherednik, *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. 61 (1984), no. 1, 35-44.

- A. Doikou, B. Rybołowicz and P. Stefanelli, Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal $\mathcal{R}$-matrices, *J. Phys. A: Math. Theor.* **57** (2024) 405203.
- V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, in Quantum groups, (Springer) Lecture Notes in Math. 1510 (1990), 1-8.
- P. I. Etingof, T. Schedler and A. V. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. 100 (1999), no. 2, 169–209.
- T. Gateva-Ivanova, M. Van den Bergh, *Semigroups of I-type*, J. Algebra 206 (1998), 97–112.
- L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comp. 86 (2017), 2519–2534.
- D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), 37-65.
- V. Lebed, L. Vendramin, *Homology of left non-degenerate set-theoretic solutions to the Yang-Baxter equation*, Adv. Math. 304 (2017), 1219–1261.
- J.-H. Lu, M. Yan and Y. Zhu, On the set-theoretical Yang-Baxter equation, Duke Math. J. 104 (2000), 1–18.

S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119 (1982), 78-88.

W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307 (2007), 153–170.

E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A 21 (1988), no. 10, 2375-2389.

A. Smoktunowicz, L. Vendramin, *On skew braces (with an appendix by N. Byott and L. Vendramin)*, J. Comb. Algebra 2 (2018), 47-86.

A. V. Soloviev, *Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation*, Math. Res. Lett. 7 (2000), 577–596.

L. D. Faddeev, N. Y. Reshetikhin, L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Algebraic analysis I, Academic Press, Boston (1988), 129-139.

C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. 19 (1967), 1312–1315.