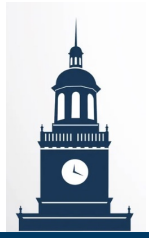

Dihedral Solutions of the set-theoretical Yang-Baxter equation

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Howard University

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Panglobal Algebra and Logic Seminar

Contents

- Various nonassociative algebras
- The set-theoretical Yang-Baxter equation
- Dihedral Solutions

A Menagerie of Nonassociative Operations

Quasigroups and loops

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  - For each $a \in Q$ the left and right multiplication maps

$$L_a^\circ : x \mapsto a \circ x \qquad (L_a^\circ)^{-1} : x \mapsto a\backslash_\circ x$$
$$R_a^\circ : x \mapsto x \circ a \qquad (R_a^\circ)^{-1} : x \mapsto x/^\circ a$$

    are invertible.

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- A *quandle* is a rack which is also idempotent: $x \circ x = x$.

Loops

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- A *loop* $(Q, +, e)$ is a quasigroup with identity.
  - $e = x \backslash_\circ x = y /^\circ y \quad \forall x, y \in Q$
- Any quasigroup $(Q, \circ, \backslash_\circ, /^\circ)$ comes with *loop isotopes* $(Q, +, e \circ e)$:

$$x + y := (R_e^\circ)^{-1}(x) \circ (L_e^\circ)(y) = (x /^\circ e) \circ (e \backslash_\circ y)$$

Some loop-quasigroup correspondences

Quasigroups:

- Involutory Latin quandle $(Q, \circ, \circ, /^\circ)$
  - $a \circ (a \circ x) = x$, i.e., $L_a^\circ = (L_a^\circ)^{-1}$
  - $a \circ (x \circ y) = (a \circ x) \circ (a \circ y)$, i.e. each $L_a^\circ \in \mathrm{Aut}(Q)$
  - $x \circ x = x$.
- Hall triple system $(Q, \circ, \circ, \circ)$
  - $a \circ (a \circ x) = x = (x \circ a) \circ a$, i.e. $L_a^\circ = (L_a^\circ)^{-1} = R_a^\circ$
  - $a \circ (x \circ y) = (a \circ x) \circ (a \circ y)$, i.e. each $L_a^\circ \in \mathrm{Aut}(Q)$
  - $x \circ x = x$

Loops:

- Uniquely 2-divisible Bruck loop
  - $x + (y + (x + z)) = (x + (y + x)) + z$
  - $-(x + y) = -x - y$, i.e. inversion is an automorphism
  - $x \mapsto x + x =: 2x$ is invertible
- Commutative Moufang loops of exponent 3 (CML_3)
  - $(x + x) + (y + z) = (x + y) + (x + z)$
  - $\langle x \rangle \cong \mathbb{Z}_{/3}$

The Set-Theoretical Yang-Baxter Equation

Yang-Baxter equation

- $R \in \mathrm{End}(V \otimes V)$ is a solution of the *quantum Yang-Baxter equation* (QYBE) if

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

  - $R_{ij} \in \mathrm{End}(V \otimes V \otimes V)$ acts on the ith and jth tensor factors.

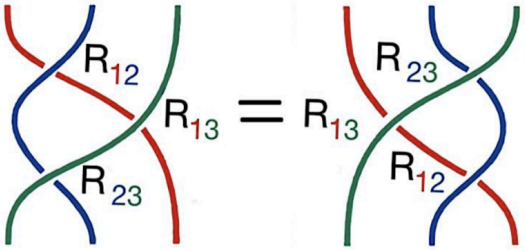
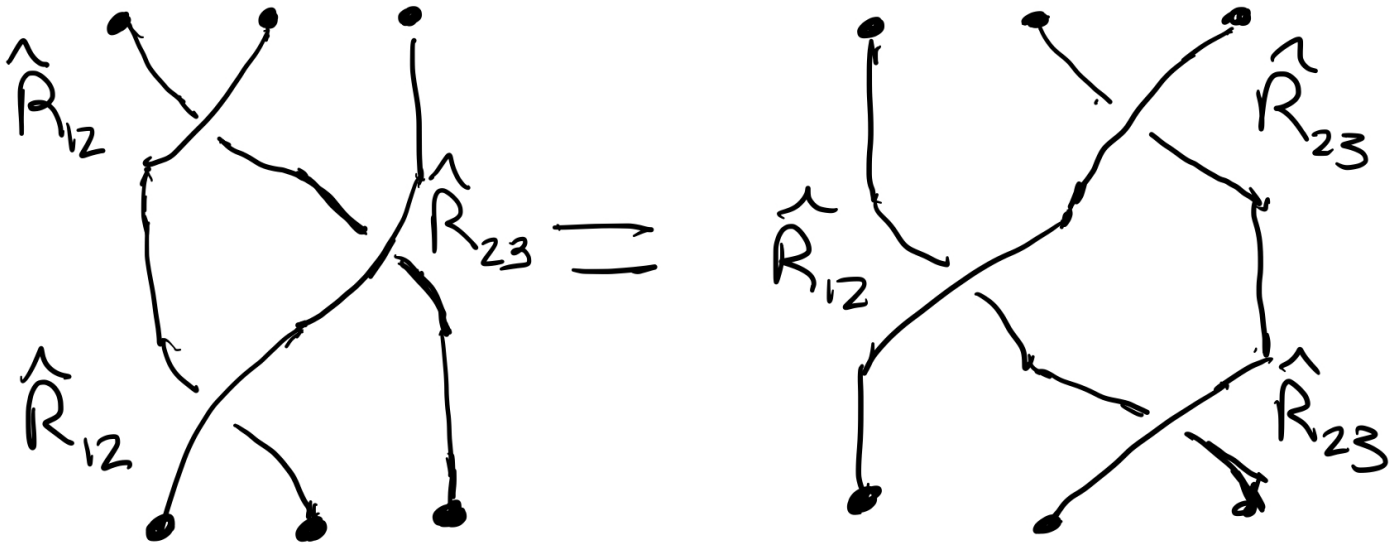


Image from Bardakov et. al., (1).

YBE and braids

- With $\tau : x \otimes y \mapsto y \otimes x$ the "flip map," R is a QYBE solution iff $\hat{R} := \tau R$ is a solution of the *braid equation*

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$



Set theoretical YBE

- Drinfeld proposed study of set-theoretical solutions (SYBE) as a simplification.
  - Let Q be a set. $r : Q^2 \to Q^2$ is a *set-theoretical solution* of the QYBE if

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- *Involutive solutions*, $r^2 = 1_{Q\times Q}$, get the most attention.
  Examples: $\tau(x, y) = (y, x)$, the trivial solution,
  $r(x, y) = (\varphi(y), \varphi^{-1}(x))$ for $\varphi \in S_Q$.

Universal algebraic approach to SYBE

- Think of potential solutions r as a pair of binary operations:

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- (Q, r) is a braided set iff $(Q, \circ, \bullet)$ satisfies:

$$\begin{aligned} x \circ (y \circ z) &= (x \circ y) \circ ((x \bullet y) \circ z) \\ (x \circ y) \bullet ((x \bullet y) \circ z) &= (x \bullet (y \circ z)) \circ (y \bullet z) \\ (x \bullet y) \bullet z &= (x \bullet (y \circ z)) \bullet (y \bullet z) \end{aligned}$$

- *Derived solutions* come from racks: $r(x, y) = (x \circ y, x)$

(Non)degeneracy

- A solution $r(x, y) = (x \circ y, x \bullet y)$ is *nondegenerate* if there is a left-quasigroup $(Q, \circ, \backslash_\circ)$ and a right quasigroup $(Q, \bullet, /^\bullet)$

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- The *diagonal* maps $S : x \mapsto x \backslash_\circ x$ and $T : x \mapsto x /^\bullet x$ show up a lot in the theory.

Dihedral solutions

Solutions from CML_3

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$$\rho : (x, y) \mapsto (-x + y, -x),$$

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  - $\rho'(x, y) = (2x - y, x)$ for Bruck loops

Properties of CML_3 solutions

Properties of $\rho(x,y) = (-x+y, -x)$ and $\rho'(x,y) = (-(x+y), x)$:

- They're not involutive, but they have order 3 ($r^3 = 1_{Q\times Q}$), and ...

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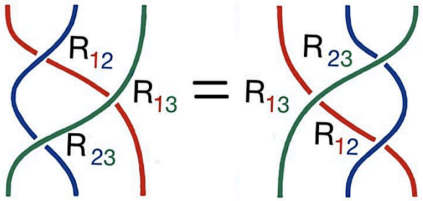

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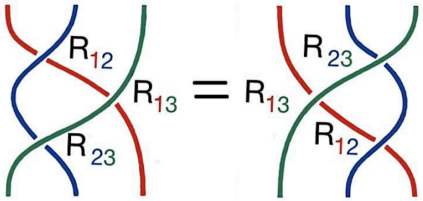

- They're *Latin*.

Dihedral/triality sets

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Theorem (N., Zamojska-Dzienio, 2025)

Let $r(x, y) = (x \circ y, x \bullet y)$ be an LBTS. Then there is a CML_3-factorization $Q = Q_e + P$ such that $r|_{Q_e}$ has the form ρ and $r|_P$ has the form ρ'.

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Lemma

For any LBDS r, we have $r(x,y) = (x \circ y, x^S)$, where $x^S = x \backslash_\circ x$.

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- Call this the variety of *LBDS*-quasigroups.

Bruck loop isotopes

- Fix $f \in Q$, and set $e = f /^{\circ} f$. If $(Q, \circ, \backslash_{\circ}, /^{\circ})$ is an LBDS-quasigroup, then $x + y = (x /^{\circ} e) \circ (e \circ x)$ is a uniquely 2-divisible Bruck loop $(Q, +, e)$ and S is an involutive automorphism of this loop.

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Theorem (N, Zamojska-Dzienio, 2025)

There is a one-to-one correspondence between isomorphism classes of LBDS-quasigroups and conjugacy classes of involutions in automorphism groups of uniquely 2-divisible Bruck loops.

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Theorem (N., Zamojska-Dzienio, 2025)

If $Sq_{/^\circ}$ is endomorphic, then we have a split exact sequence

$$\{e\} \to Q_e \to Q \to Sq_{/^\circ}(Q) \to \{e\}$$

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Let p be any odd prime. Up to isomorphism there are...

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- 18 LBTS of order $\leq 3^4$.

References I

[1] V. Bardakov, M. Elhamdadi, M. Singh, "Yang Baxter equation and related algebraic structures," arXiv:2506.23175.

Thank you for your attention

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