Clones over finite sets up to minor-equivalence

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joint work with M. Bodirsky and D. Zhuk

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TCS4F THEORETICAL COMPUTER SCIENTISTS FOR FUTURE

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- $\mathcal{P}_2 := \langle \emptyset \rangle$ (*the clone of all projections on* $\{0,1\}$);
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- $\langle 0 \rangle$ *(the clone generated by the constant operation 0).*

A Galois connection for clones

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An operation $f\colon A^n \to A$ preserves a k-ary relation R on A if

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & & a_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & a_{k,2} & & a_{k,n} \\ \in & \in & & \in \\ R & R & & R \end{array}$$

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- $\mathrm{Inv}(F) = \{R \mid R$ *is invariant under every operation in* $F\}$.

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We call $\{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$ *the relation defined by* ϕ.

If ϕ is primitive positive, then this relation is said to be pp-definable in $\mathbb{A}$.

Theorem (Geiger; Bodnarčuk, Kalužnin, Kotov, Romov)

A relation R *is pp-definable in* $\mathbb{A} \iff R$ *is in* $\mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$.

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Theorem (Geiger; Bodnarčuk, Kalužnin, Kotov, Romov)

If F is a set of operations on a finite domain, then $\mathrm{Pol}(\mathrm{Inv}(F)) = \langle F \rangle$.

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Corollary

All clones over a finite n-element set form a lattice $\mathfrak{L}_n$ *under inclusion.*

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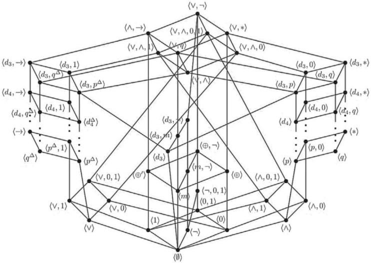

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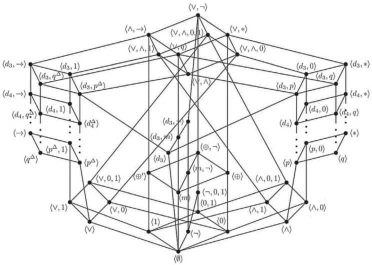

Corollary

- $\mathbb{A}$, $\mathbb{B}$: *relational structures on the same finite universe A,*
- $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$ *and* $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$.

Then $\mathbb{A}$ *pp-defines* $\mathbb{B} \iff \mathcal{A} \subseteq \mathcal{B}$.

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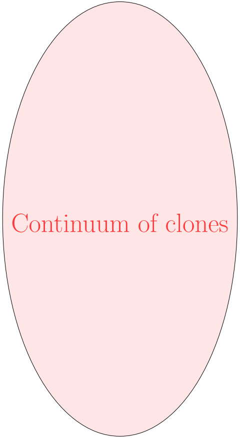

☹ There exists a continuum of clones over $\{0, 1, 2\}$ (Yanov, Muchnik '59).

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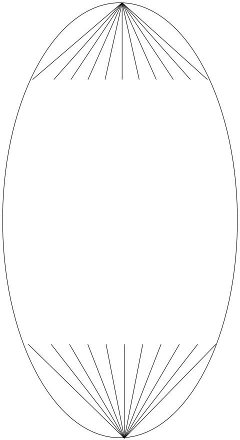

☺ Description of all maximal and minimal clones.
(Jablonskij '54; Csákány '83)

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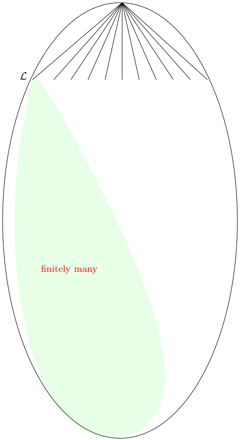

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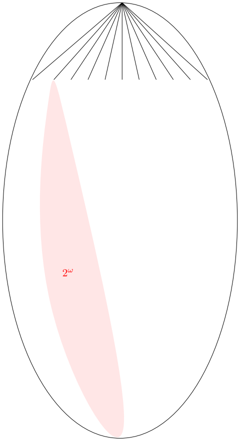

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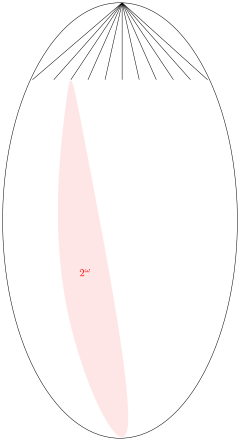

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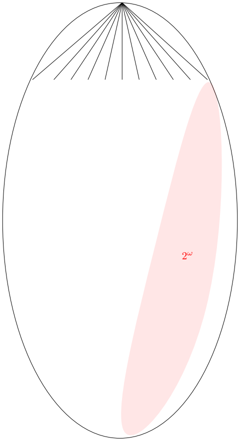

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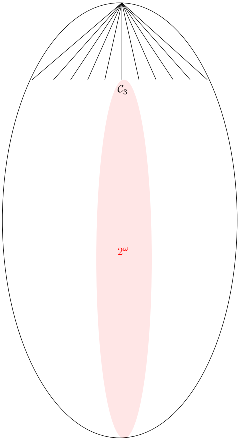

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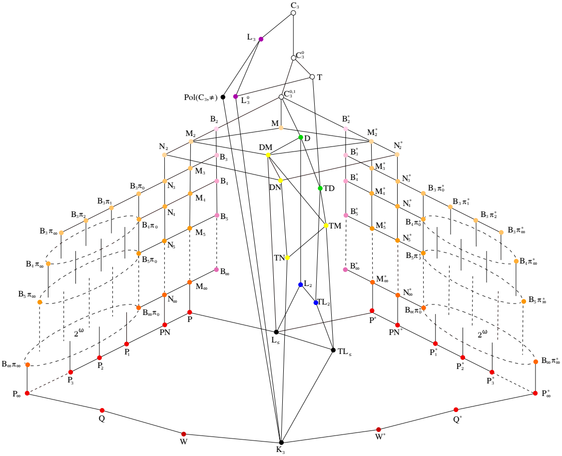

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A new order

- What we want: clones that *share similar properties* to be in the same class.

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Definition

- τ: *set of function symbols;*
  A minor identity (height 1 identity) is an identity of the form

$$f(x_1, \dots, x_n) \approx g(y_1, \dots, y_m)$$

  where $f, g \in \tau$ *and* $x_1, \dots, x_n, y_1, \dots, y_m$ *are not necessarily distinct.*
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- Some examples:

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$$f(f(x, y), z) \approx f(x, f(y, z)) \ ✖$$

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We say that F satisfies Σ ($F \models \Sigma$) if there is a map ξ assigning to each function symbol occurring in Σ an operation in F of the same arity, such that if $p \approx q$ is in Σ, then $\xi(p) = \xi(q)$.

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Theorem (Birkhoff, 1935)

Let $\mathcal{A}$, $\mathcal{B}$ be clones over finite sets. The following are equivalent:

1. *There exists a clone homomorphism from $\mathcal{A}$ to $\mathcal{B}$;*
2. *$\mathcal{B} \in \boldsymbol{EHSP}_{\mathrm{fin}}(\mathcal{A})$.*

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Let $\mathcal{A}$, $\mathcal{B}$ be clones over finite sets. The following are equivalent:

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Motivation: CSP

- $\mathbb{A}$, $\mathbb{B}$: finite relational structure with finite signature;
- a given primitive positive τ-sentence Φ.

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CSP($\mathbb{A}$) *is the computational problem of deciding whether* Φ *is true in* $\mathbb{A}$.

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$\mathbb{B}$ *is a pp-power of* $\mathbb{A}$ *if it is isomorphic to a structure with domain* A^n*, where* $n \geq 1$*, whose relations are pp-definable from* $\mathbb{A}$.

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Theorem (Barto, Opršal, Pinsker ´15)

If $\mathbb{A}$ *pp-constructs* $\mathbb{B}$, *then there exist a log-space reduction from* CSP($\mathbb{B}$) *to* CSP($\mathbb{A}$).

Motivation: CSP

Theorem (Barto, Opršal, Pinsker, 2015)

Let $\mathbb{A}$, $\mathbb{B}$ *be finite relational structures;* $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$. *TFAE:*

1. *There exists a minor-preserving map from* $\mathcal{A}$ *to* $\mathcal{B}$ *(*$\mathcal{A} \leq_{\mathrm{m}} \mathcal{B}$*);*
2. $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ *(*$\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$*);*
3. *if* $\mathcal{A}$ *satisfies a minor condition* Σ, *then* $\mathcal{B} \models \Sigma$.

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Great achievement: CSP Dichotomy Theorem!

- positive solution to the Feder-Vardi conjecture, open since 1998;
- new algebraic theories for finite algebras
  (Absorption, Bulatov-edges, strong subalgebras...)

Theorem (Bulatov 2017; Zhuk 2017)

If there is no minor-preserving map from $\mathcal{A}$ *to* $\mathcal{P}_2$*, then* $\mathrm{CSP}(\mathbb{A})$ *is in P. Otherwise,* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete*

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If $\mathbb{A}$ does not pp-construct $\mathbb{K}_3 = (\{0,1,2\}; \neq)$, then $\mathrm{CSP}(\mathbb{A})$ is in P. Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete

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Great achievement: CSP Dichotomy Theorem!

- positive solution to the Feder-Vardi conjecture, open since 1998;
- new algebraic theories for finite algebras (Absorption, Bulatov-edges, strong subalgebras,...)

Theorem (Bulatov 2017; Zhuk 2017)

If $\mathcal{A}$ satisfies a non-trivial minor condition, then $\mathrm{CSP}(\mathbb{A})$ is in P. Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete

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$$\mathfrak{P}_{\mathrm{fin}} := \left(\overline{\mathcal{C}} \mid \mathcal{C} \textit{ is a clone over some finite set}; \leq_{\mathrm{m}}\right)$$
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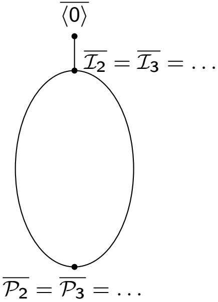

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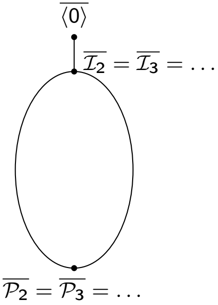

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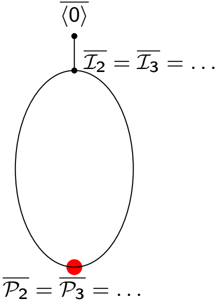
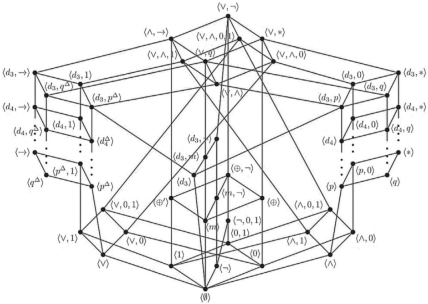


How powerful are minor-preserving maps?



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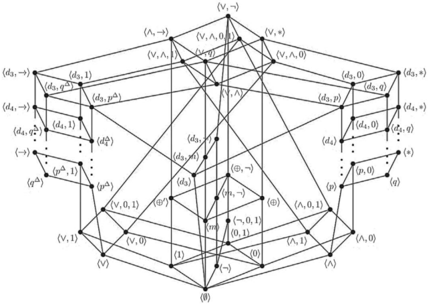

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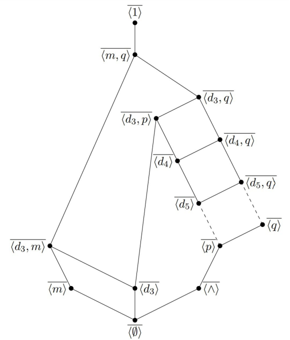

$\mathcal{P}_2$
(Bodirsky, V.)

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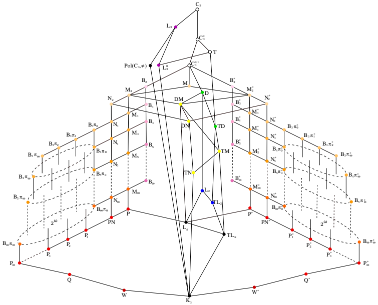

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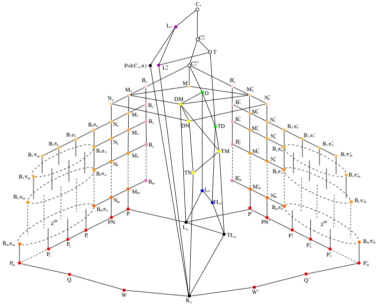

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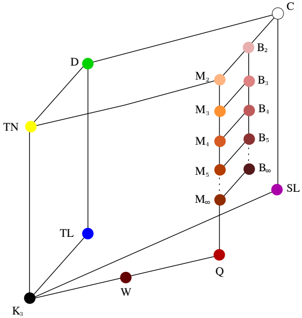

Clones of self-dual operations modulo minor-equivalence (Bodirsky, V., Zhuk)

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$$h((a_1, b_1), \dots, (a_n, b_n)) := (f(a_1, \dots, a_n), g(b_1, \dots, b_n))$$

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  where $a_i \in A$ and $b_i \in B$ for every $i \in \{1, \dots, n\}$.
- $\Gamma^{\mathbb{A} \otimes \mathbb{B}} := \mathrm{Inv}(\{(f, g) \mid f \in \mathrm{Pol}(\mathbb{A}), g \in \mathrm{Pol}(\mathbb{B})\})$; we define

$$\mathbb{A} \otimes \mathbb{B} := (A \times B; \Gamma^{\mathbb{A} \otimes \mathbb{B}}).$$

Proposition

$\overline{\mathbb{A} \otimes \mathbb{B}}$ *is the greatest lower bound of* $\overline{\mathbb{A}}$ *and* $\overline{\mathbb{B}}$.

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- from $(\star)$ it follows that $\mathbb{A} \models c(x_1, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$, for some prime $p > |A|$ ($\mathbb{A} \models \Sigma_p$);

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- take $\mathbb{B} = \mathbb{A} \otimes \mathbb{C}_p$
  1. $\mathbb{B} \not\models \Sigma_p \implies \overline{\mathbb{B}} <_{\mathrm{Con}} \overline{\mathbb{A}}$
  2. $\mathbb{B} \models \Sigma_q$, for some $q > p \cdot |A| \implies \overline{\mathbb{B}} \neq \overline{\mathbb{K}_3}$.

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Where to look:

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  Barto, Brady, Bulatov, Kozik, and Zhuk (2021)

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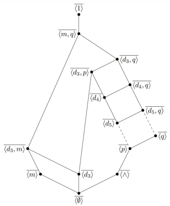
     Atoms in $\mathfrak{P}_2$: $\overline{\langle \vee \rangle} = \overline{\langle \wedge \rangle}$, $\overline{\langle m \rangle}$, $\overline{\langle d_3 \rangle}$.



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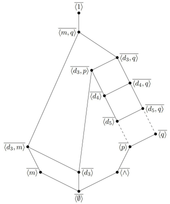

  2. n=3 False! $\implies$ "Atoms are better than Minimal Taylor"
     (Barto, Brady, V., Zhuk)

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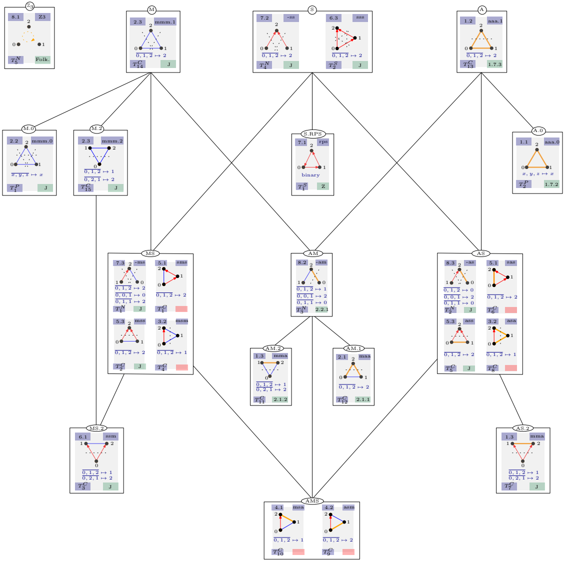

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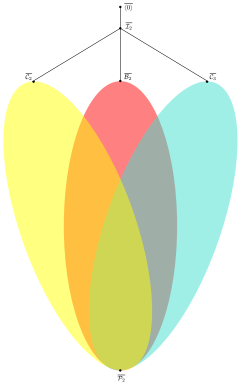
Theorem (V., Zhuk)

$\mathfrak{P}_3$ has exactly three submaximal elements: $\overline{\mathcal{C}_2}$, $\overline{\mathcal{C}_3}$, and $\overline{\mathcal{B}_2}$



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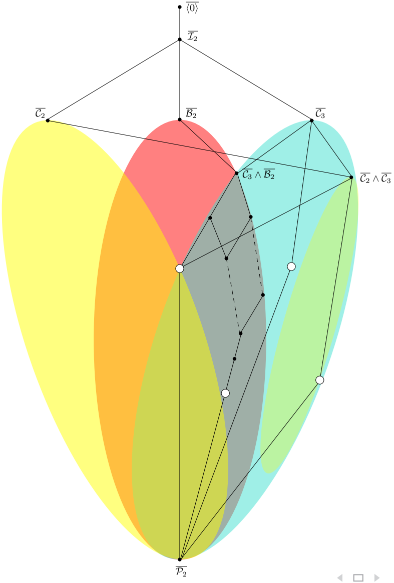

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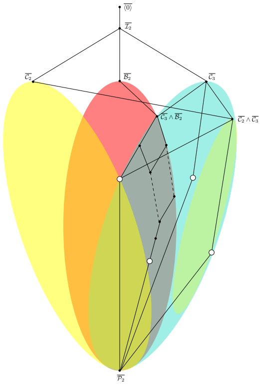

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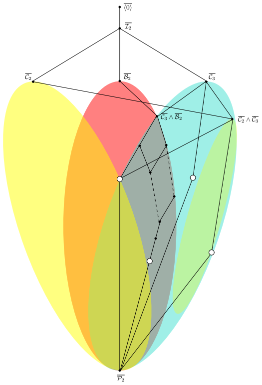

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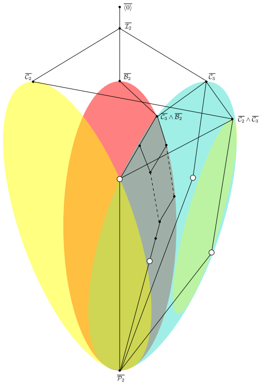

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- Below $\overline{\mathcal{C}_2}$: Mild! ☺
- Below $\overline{\mathcal{B}_2}$: Wild! (potentially 2^{ω} elements) ☹

Ongoing

1. Cardinality of $\mathfrak{P}_{\mathrm{fin}}$: We know where to look (again below $\overline{\mathbb{B}_2}$).

Theorem (Aichinger, Mayr, McKenzie 2014)

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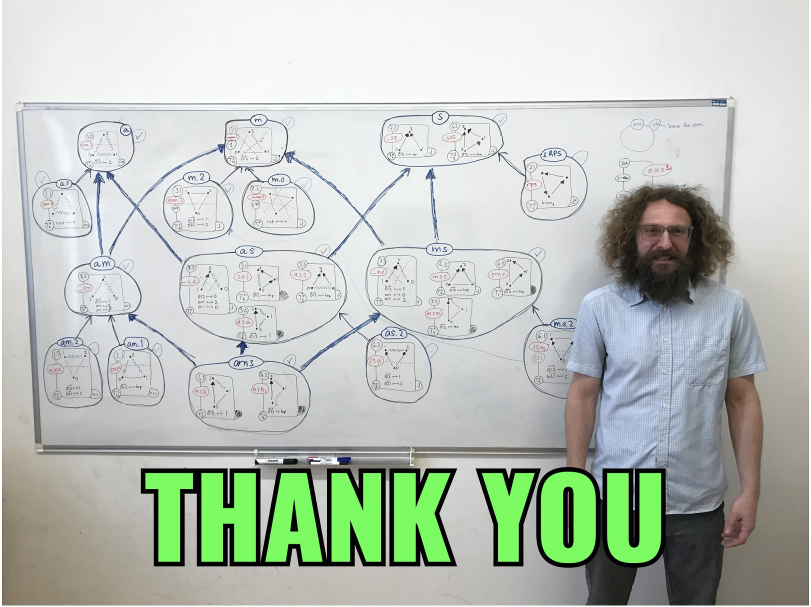
3. Mal'cev clones over $\{0, 1, 2\}$ up to minor-equivalence (Fioravanti, Rossi, V.).


THANK YOU