Talk #8: Labeled congruence lattices

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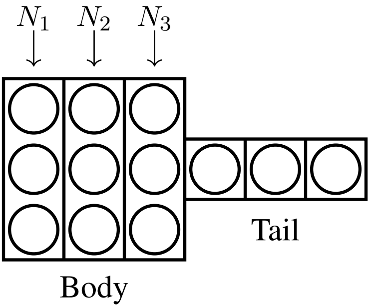

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Since $(r, s) \notin \alpha$, at least one link is not an α-link and necessarily it is of the form $\{g(p), g(q)\}$ for some polynomial g such that $g(A) \subseteq f(A) = V$.

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Thus, the "type" of $\langle\alpha, \beta\rangle$ is well defined, and we write $\alpha \overset{\mathbf{i}}{\prec} \beta$ for $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ to indicate it.

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Stage 2. $\alpha \vee \beta = \alpha \circ_n \beta = \alpha \circ \beta \circ \alpha \circ \cdots$ for sufficiently large n.

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Stage 3.

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Choose $\sigma \in \mathrm{Con}(\mathbf{A}|_U) \subseteq \mathrm{Rel}(\mathbf{A}|_U)$. There exists $\alpha \in \mathrm{Rel}(\mathbf{A})$ such that $\alpha|_U = \sigma$.

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Choose n so that $\gamma = \circ_n \beta$ be the transitive closure of β.

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Choose n so that $\gamma = \circ_n\, \beta$ be the transitive closure of β.

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We have $\gamma \in \mathrm{Con}(\mathbf{A})$ and $\gamma|_U = \sigma$.

Stage 4. ($|_U$ is label-preserving)

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Multiminimal algebras

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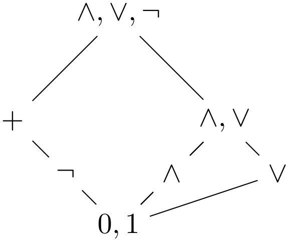

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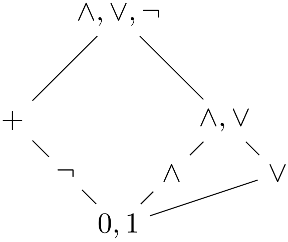

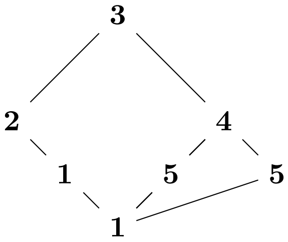

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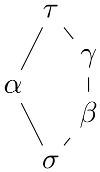

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2. The members of a locally finite variety $\mathcal{V}$ have modular congruence lattices if and only if $\mathcal{V}$ omits types **1** and **5** and all minimal sets have empty tails.
3. If $\mathcal{V}$ is a locally finite variety that omits types **1**, **2** and **5**, then there is a lattice identity that holds in $\mathrm{Con}(\mathbf{A})$ for every $\mathbf{A} \in \mathcal{V}$ that is strong enough to imply that all algebras have join-semidistributive congruences lattices.
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The other part of the explanation is that the satisfaction of congruence identities can be characterized by idempotent linear Maltsev conditions, and these restrict well to minimal sets.

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