Talk #7: $\langle \alpha, \beta \rangle$-minimal algebras, 2

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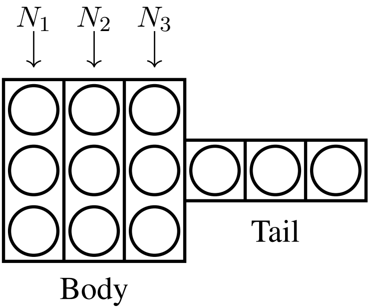

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$m(x,y)$	O	I	T
O	O	O	?
I	O	I	?
T	?	?	?

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Let's explain why the neutral coset I is a singleton α-class (Part 1).

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Proof of the refinement

Let's explain why the neutral coset I is a singleton α-class (Part 1).

Start with any $f(x,y)$ that induces a semilattice operation on $\mathbf{A}|_B/\alpha|_B$. The polynomial $\pi(x) := f(x,x)$ does not collapse β into α ($\pi(I) \subseteq I$ and $\pi(O) \subseteq O$), so π is a permutation of $\mathbf{A}$ (which induces the identity on $\mathbf{A}|_B/\alpha|_B$). Let $g(x,y) = \pi^{-1}(f(x,y))$. Then $g(x,x) = \pi^{-1}(\pi(x)) = x$ and g induces the same semilattice operation as f on $\mathbf{A}|_B/\alpha|_B$. Iterate $g(x,y)$ in its first variable to obtain $h(x,y)$ satisfying $h(h(x,y),y) = h(x,y)$ on $\mathbf{A}$. f, g and h induce the same operation on $\mathbf{A}|_B/\alpha|_B$, and h and g are both idempotent polynomials, but now $h(x,u) = x$ for any $x \in A$ and $u \in I$. Next, iterate $h(x,y)$ in its second variable to obtain $m(x,y)$ satisfying $m(x,m(x,y)) = m(x,y)$ on $\mathbf{A}$. f, g, h and m induce the same operation on $\mathbf{A}|_B/\alpha|_B$. g, h, m are idempotent. h and m both satisfy $h(x,u) = x = m(x,u)$ for $x \in A$ and $u \in I$, but I claim that $m(x,y)$ also satisfies $m(u,x) = x$ for such u, x. Finally, if $u, u' \in I$, then $u = m(u,u') = u'$ by the properties of m. $\square$

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3. If $t \in A \setminus B$ belongs to the tail, then $m(0, t) \equiv_\alpha m(t, 0) \equiv_\alpha t$ and $j(1, t) \equiv_\alpha j(t, 1) \equiv_\alpha t$.

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$m(x, y)$ is called a **pseudo-meet** polynomial and $j(x, y)$ is called a **pseudo-join** polynomial of $\mathbf{A}$ with respect to $\langle\alpha, \beta\rangle$ if they have these properties.

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Observation. If N_1, N_2 are both traces of the same $\langle\alpha, \beta\rangle$-minimal algebra, then N_1 and N_2 are polynomially isomorphic, and therefore $\mathbf{A}|_{N_1}$ and$\mathbf{A}|_{N_2}$ have isomorphic polynomial clones.

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Hence there is a Maltsev chain connecting a_2 to b_2 that consists of α-links and links that are polynomial images $(p(a_1), p(b_1))$ of the pair (a_1, b_1).

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Hence there is a Maltsev chain connecting a_2 to b_2 that consists of α-links and links that are polynomial images $(p(a_1), p(b_1))$ of the pair (a_1, b_1). Each link of the chain is a β-link, so the the chain lies entirely inside N_2.

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This provides many examples of $\langle\alpha, \beta\rangle$-minimal algebras of Type **1**.

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Condensing: If $\mathbf{A}$ is $\langle\alpha, \beta\rangle$-minimal of Type $\mathbf{2}$, N is a trace of $\mathbf{A}$,

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Type 2

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   2. $\mathbf{A}|_B$ is nilpotent and $\mathbf{A}|_N/\alpha|_N$ is abelian when the type is **2**.
   3. $\mathbf{A}|_N/\alpha|_N$ is strongly abelian when the type is **1**.