Talk #6: $\langle\alpha, \beta\rangle$-minimal algebras

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The Target Theorem

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Visual Target

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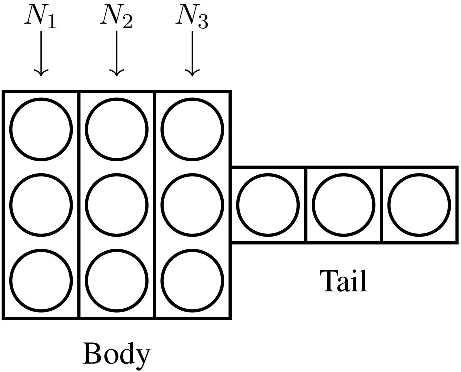

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Claim 1. If $c \in A \setminus \{a\}$, then $p(x, c)$ is collapsing.
Assume not. Choose elements $a' \neq a$ and $b' \neq b$ such that $\{a, a'\}$ is contained in a trace and $\{b, b'\}$ is contained in a trace.

Proof of the Twin Lemma

It suffice to prove the Twin Lemma in the case where $\alpha = 0$.

Assume that $p'_{\bar{a}}(x)$ and $p'_{\bar{b}}(x)$ are body twins of different character. We may assume that $|\bar{a}| = 1 = |\bar{b}|$. Thus, we have $p'(x, y)$ and $a, b \in B$ such that $p'(x, a)$ is a permutation of A while $p'(\beta, b) \subseteq \alpha$.

Iterate $p'(x, y)$ in its first variable to obtain $p(x, y) = p'(p'(\cdots p'(p'(x, y), y) \cdots, y), y)$ satisfying $p(p(x, y), y) = p(x, y)$ on $\mathbf{A}$. $p(x, a)$ and $p(x, b)$ are body twins where the first is a permutation and the second is collapsing. Since $p(x, a)$ is an idempotent permutation of $\mathbf{A}$ we have $p(x, a) = x$ on $\mathbf{A}$. Since $p(x, b)$ is collapsing on $\mathbf{A}$, $p(x, b)$ is constant on β-classes. ($p(\beta, b) \subseteq 0$.)

Claim 1. If $c \in A \setminus \{a\}$, then $p(x, c)$ is collapsing.
Assume not. Choose elements $a' \neq a$ and $b' \neq b$ such that $\{a, a'\}$ is contained in a trace and $\{b, b'\}$ is contained in a trace. (Here, using that $a, b \in B$)

A fragment of the table for $p(x, y)$.

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p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

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p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.

A fragment of the table for $p(x, y)$.

p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

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p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.
2. All rows are collapsing.

A fragment of the table for $p(x, y)$.

p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

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A fragment of the table for $p(x, y)$.

p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.
2. All rows are collapsing.
3. $q(x) = p(x, x)$ is neither permutational nor collapsing.

A fragment of the table for $p(x, y)$.

p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

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a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.
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3. $q(x) = p(x, x)$ is neither permutational nor collapsing. ($q(x) = x$ on a/β and q is constant on b/β)

A fragment of the table for $p(x, y)$.

p	a	a'	$\cdots$	b	b'	$\cdots$	c	$\cdots$
a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.
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a	a	a	$\cdots$			$\cdots$	a	$\cdots$
a'	a'	a'	$\cdots$			$\cdots$	a'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
b	b	b	$\cdots$	u	u	$\cdots$	b	$\cdots$
b'	b'	b'	$\cdots$	u	u	$\cdots$	b'	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$
c	c	c	$\cdots$			$\cdots$	c	$\cdots$
$\vdots$			$\ldots$			$\ldots$		$\ldots$

1. a- and c-columns are the identity.
2. All rows are collapsing.
3. $q(x) = p(x, x)$ is neither permutational nor collapsing. ($q(x) = x$ on a/β and q is constant on b/β) $\Rightarrow\Leftarrow$

   $\square$ (Claim 1)

Proof of the Twin Lemma

Proof of the Twin Lemma

Claim 2.

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Claim 2. $|B| = 2$.

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Proof of the Twin Lemma

Claim 2. $|B| = 2$.
Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

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Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

Subclaim.

Proof of the Twin Lemma

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Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

Subclaim. If $(u, v) \in \beta|_B$, then $p(u, y)$ and $p(v, y)$ have the same character.

Proof of the Twin Lemma

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Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

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Proof of Subclaim.

Proof of the Twin Lemma

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Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

Subclaim. If $(u, v) \in \beta|_B$, then $p(u, y)$ and $p(v, y)$ have the same character.

Proof of Subclaim. Assume that $p(u, y)$ is collapsing.

Proof of the Twin Lemma

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Subclaim. If $(u, v) \in \beta|_B$, then $p(u, y)$ and $p(v, y)$ have the same character.

Proof of Subclaim. Assume that $p(u, y)$ is collapsing. Then $p(v, b) = p(u, b) = p(u, c) = p(v, c)$, so $p(v, y)$ is collapsing.

Proof of the Twin Lemma

Claim 2. $|B| = 2$.
Assume not. If $|B| > 2$, then B contains more than one trace or B contains one trace with more than 2-elements. Therefore, we may assume that there exist distinct $a, b, c \in B$ such that $(b, c) \in \beta$, $p(x, a) = x$, and such that $p(x, b)$ and $p(x, c)$ are collapsing.

Subclaim. If $(u, v) \in \beta|_B$, then $p(u, y)$ and $p(v, y)$ have the same character.

Proof of Subclaim. Assume that $p(u, y)$ is collapsing. Then $p(v, b) = p(u, b) = p(u, c) = p(v, c)$, so $p(v, y)$ is collapsing. □ (Subclaim)

Proof of the Twin Lemma

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To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x,a) = x$ we have $p(a,a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a,a) = a \in a/\beta = N$.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$.

Proof of the Twin Lemma

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$p\|_N$	a	a'	a''	$\cdots$
a	a	r	s	$\cdots$
a'	a'	r	s	
a''	a''	r	s	
$\vdots$				$\ddots$

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1. All columns are constant except the a-column.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

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a	a	r	s	$\cdots$
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a''	a''	r	s	
$\vdots$				$\ddots$

1. All columns are constant except the a-column.
2. At most 1 row can be a permutation.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

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a	a	r	s	$\cdots$
a'	a'	r	s	
a''	a''	r	s	
$\vdots$				$\ddots$

1. All columns are constant except the a-column.
2. At most 1 row can be a permutation.
3. At least 1 row is collapsing.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

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1. All columns are constant except the a-column.
2. At most 1 row can be a permutation.
3. At least 1 row is collapsing.
4. By the Subclaim, all rows have the same character.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

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1. All columns are constant except the a-column.
2. At most 1 row can be a permutation.
3. At least 1 row is collapsing.
4. By the Subclaim, all rows have the same character.
5. At least one column is constant, so $p|_N$ is constant.

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To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

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2. At most 1 row can be a permutation.
3. At least 1 row is collapsing.
4. By the Subclaim, all rows have the same character.
5. At least one column is constant, so $p|_N$ is constant.
6. But it is not.

Proof of the Twin Lemma

To finish Claim 2, notice that since $p(x, a) = x$ we have $p(a, a) = a$. If $N := a/\beta$ and $a', a'' \in N$, then $p(a', a'') \equiv_\beta p(a, a) = a \in a/\beta = N$. That is, $p(N, N) \subseteq N$. Thus, $p|_N(x, y)$ is a binary operation on N.

$p\|_N$	a	a'	a''	$\cdots$
a	a	r	s	$\cdots$
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$p\|_N$	a	a'	a''	$\cdots$
a	a	r	s	$\cdots$
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3. At least 1 row is collapsing.
4. By the Subclaim, all rows have the same character.
5. At least one column is constant, so $p|_N$ is constant.
6. But it is not. $\square$ (Claim 2)

Proof of the Twin Lemma

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Necessarily $B = \{a, b\} = a/\beta = N$. We know that $p(x, a) = x$ and that $p(x, b)$ is collapsing.

Proof of the Twin Lemma

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$p\|_B$	a	b
a	a	b
b	b	b

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In the first case, $p(x, y)$ is a semilattice operation on B with absorbing element b. In the second case, $p(x, p(b, y))$ is a semilattice operation on B with absorbing element a.

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In the first case, $p(x, y)$ is a semilattice operation on B with absorbing element b. In the second case, $p(x, p(b, y))$ is a semilattice operation on B with absorbing element a. $\square$ (Twin Lemma)