Talk #4: Classification

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Question. How must simpler is $\mathbf{A}|_U$ compared to $\mathbf{A}$ after simplifying as much as possible?

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Definition. $M_{\mathbf{A}}(\alpha, \beta)$ is the set of $\langle \alpha, \beta \rangle$-minimal sets. (= The set of minimal images $f(A)$ for $f \in C_1(\mathbf{A})$ where $f(\beta) \not\subseteq \alpha$.)

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1. Each $\langle \alpha, \beta \rangle$-minimal set is a neighborhood. (Covering intervals are "tame".)
2. All members of $M_{\mathbf{A}}(\alpha, \beta)$ are isomorphic.
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4. (Separation) If $(x, y) \in \beta - \alpha$ and $U \in M_{\mathbf{A}}(\alpha, \beta)$, then $\exists f \in C_1(\mathbf{A})$ such that $(f(x), f(y)) \in \beta|_U - \alpha$.
5. (Density/Connectedness) If N is a $\beta|_U$-class of $U \in M_{\mathbf{A}}(\alpha, \beta)$, then the compatible relation generated by $\alpha \cup (N \times N)$ has transitive closure equal to β.

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0	0	1	2	3	4
1	1	0	3	2	4
2	2	3	0	1	4
3	3	2	1	0	4
4	4	4	4	4	4

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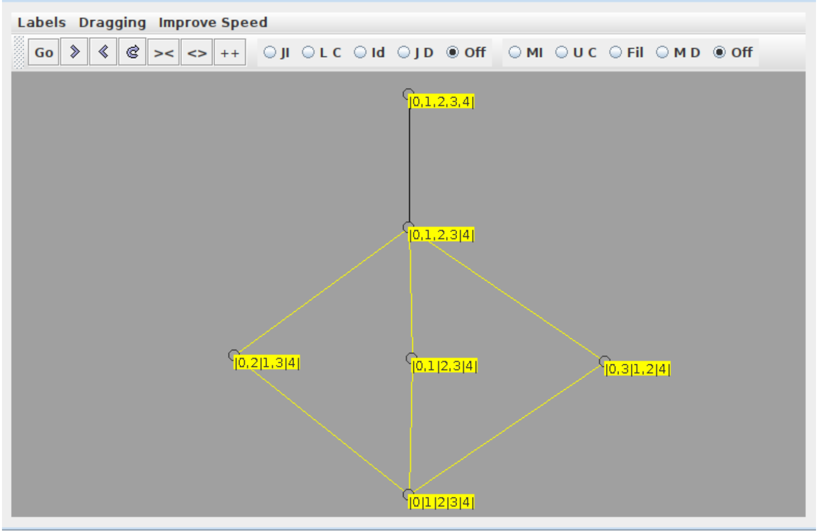

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This example is the polynomial expansion of the semigroup obtained from the Klein group by adding an absorbing element. One can create similar examples with more interesting bodies by taking the polynomial expansion of the semigroup obtained from the any p-group (=body) by adding an entire semilattice of absorbing elements (=tail).