Talk #3: Covers

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7. $\mathbf{A}|_U = (\mathbf{A}^{\perp}|_U)^{\perp} = \langle U; e(\mathcal{C}) \rangle$ where $e(\mathcal{C}) = \{et \mid t \in \mathcal{C}\} = \bigcup_n \{t \in C_n \mid t(U^n) \subseteq U\}$.

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Definition. A *cover* of $\mathbf{A}$ is a set $\mathcal{U}$ of neighborhoods for which

$$\bigwedge_{U \in \mathcal{U}} S|_U = T|_U \implies S = T$$

for all $S, T \in \mathcal{R}$.

Picture

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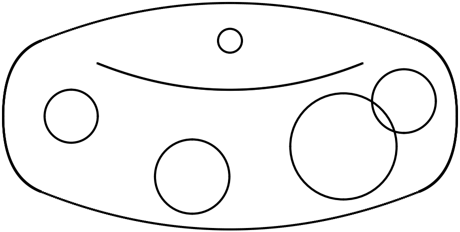

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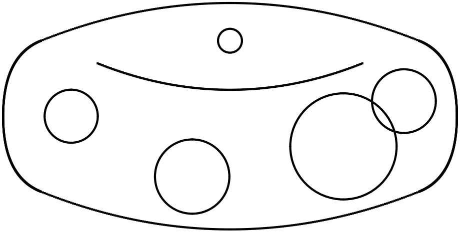

$\mathcal{U}$ is a cover if the collection of relational clone homomorphisms $|_U : \mathcal{R} \to \mathcal{R}|_U$ is jointly 1-1.

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$$\lambda(e_1\rho_1(x), \ldots, e_m\rho_m(x)) = x$$

where $e_i(A) \in \mathcal{U}$ for all i and $\lambda, \rho_i \in \mathcal{C}$.

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This provides an avenue to reduce the study of modules to the study of modules over local rings.

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Refinements

From now on $\mathbf{A}$ is finite. If $\mathcal{U} = \{U_1, \ldots, U_m\}$ covers $\mathbf{A}$, then $\mathbf{A}$ is "reconstructible" from $\mathbf{A}|_{U_1}, \ldots, \mathbf{A}|_{U_m}$. We may try to further decompose each $\mathbf{A}|_{U_i}$ by the same method. This leads to the concept of a refinement of a cover.

Definition. The set $\mathcal{V}$ of neighborhoods *covers* the neighborhood U if

$$\bigwedge_{V \in \mathcal{V}} S|_V = T|_V \Longrightarrow S|_U = T|_U$$

for all $S, T \in \mathcal{R}$.

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