The Structure of Finite Algebras: Tame Congruence Theory #2

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[I will typically write "$f \perp R$" for "$(f, R) \in \text{compatibility}$". For $\mathcal{F} \subseteq \mathrm{Op}$, I will write $\mathcal{F}^{\perp}$ for $\{R \in \mathrm{Rel} \mid (\forall f \in \mathcal{F})(f \perp R)\}$ and $\mathcal{R}^{\perp}$ for $\{f \in \mathrm{Op} \mid (\forall R \in \mathcal{R})(f \perp R)\}$.]

Picture

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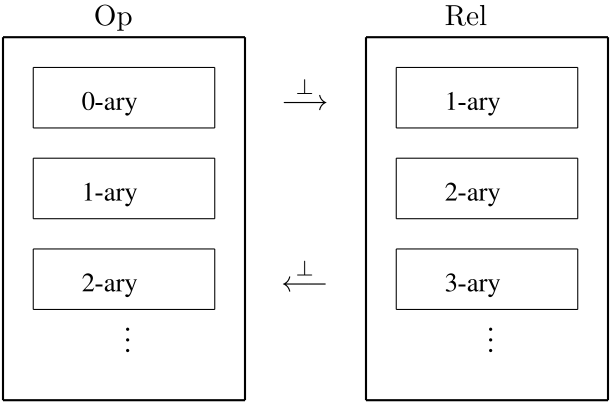

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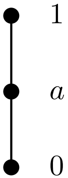

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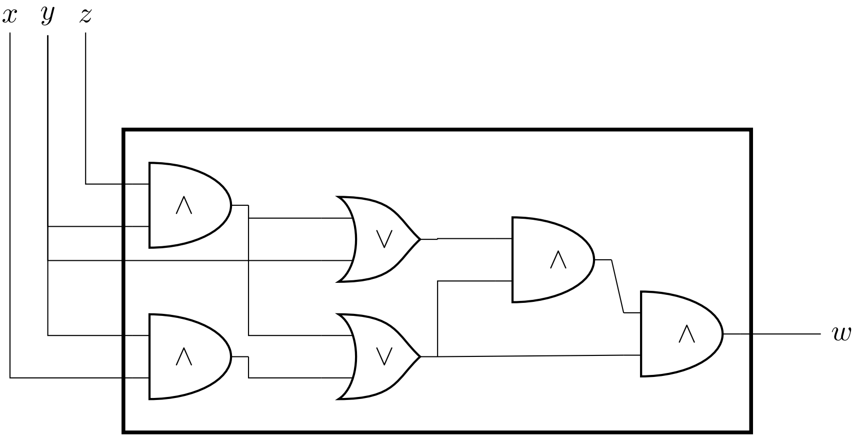

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These relations may be considered to be ‘ ‘fundamental’ constraints’ on computation in $\mathbf{A}$.

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be the A-ary relation consisting of graphs of unary operations of the non-indexed algebra $\mathbf{A}$.

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$$T = \{(t(a_i))_{i<|A|} \mid t \in C_1(\mathbf{A})\}$$

be the A-ary relation consisting of graphs of unary operations of the non-indexed algebra $\mathbf{A}$. Let $proj_U$ be projection onto the coordinates in U. Note that $proj_U(T)|_U$ contains the graph of the identity function on U. If $proj_U(T)|_U = proj_U(T|_U)$, then $proj_U(T|_U)$ also contains the graph of the identity function on U. Thus T must contain the graph of a unary term whose range is in U and which is the identity function on U; i.e., an idempotent unary term with range U. $\square$

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$e(\mathcal{C}) = \{et \mid t \in \mathcal{C}\} = \bigcup_n \{t \in C_n \mid t(U^n) \subseteq U\}$.

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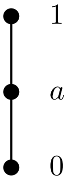

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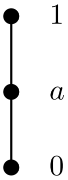

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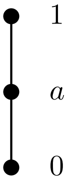

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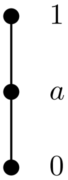

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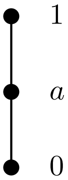

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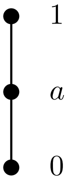

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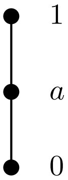

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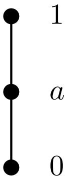

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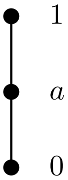

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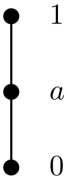

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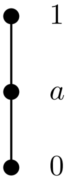

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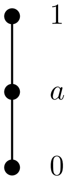

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$$\mathbf{A}|_U, \mathbf{A}|_V, \mathbf{A}|_{\{0\}}, \mathbf{A}|_{\{a\}}, \mathbf{A}|_{\{1\}}.$$

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Definition. A *cover* of $\mathbf{A}$ is a set $\mathcal{U}$ of neighborhoods for which

$$\bigwedge_{U \in \mathcal{U}} S|_U = T|_U \Longrightarrow S = T$$

for all $S, T \in \mathcal{R}$.

Globalization picture

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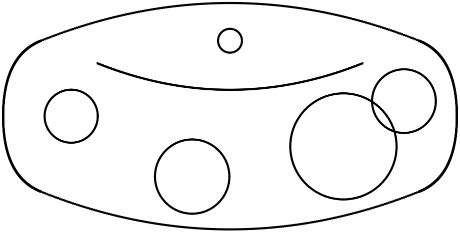

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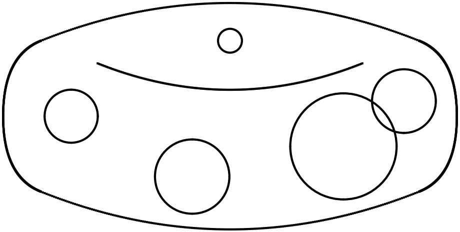

So, $\mathcal{U}$ is a cover if the sequence of relational clone homomorphisms $\rho_U : \mathcal{R} \to \mathcal{R}|_U$ is jointly 1-1.

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