Talk #10: Interpretations

Semilattices are complex enough to encode graphs

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The graph

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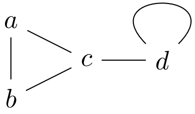

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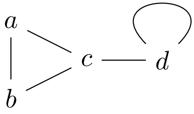

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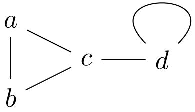

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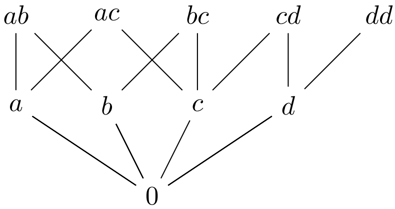

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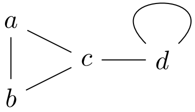

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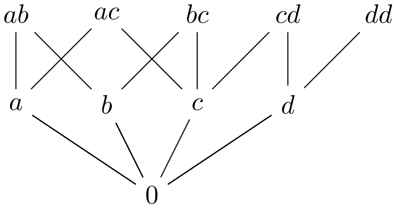

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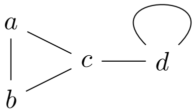

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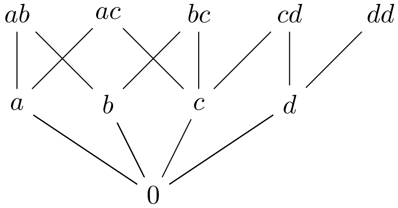

Vertices = atoms. Edge between $x, y \in V$ represented by a height-2 element that dominates both x and y. This IS a semilattice order, provided …

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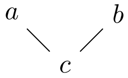

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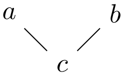

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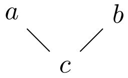

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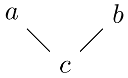

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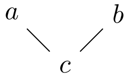

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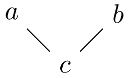

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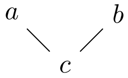

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& & G_c & &
\end{tikzcd}$$

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(iii) $R(h(\mathbf{u}_1), \ldots, h(\mathbf{u}_k)) \iff \rho_R(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b})$.

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Moreover, it is impossible to interpret the class of graphs into any of $\mathcal{S}, \mathcal{A}, \mathcal{D}$.

Local structure in decidable varieties, 4

Theorem. Let $\mathcal{V}$ be a locally finite variety. If it is impossible to interpret the class of graphs into $\mathcal{V}$, then there is a unique decomposition

$$\mathcal{V} = \mathcal{S} \otimes \mathcal{A} \otimes \mathcal{D}$$

where

1. $\mathcal{S}$ is a type-**1** strongly abelian variety. (The structure of $\mathcal{S}$ is fully known. $\mathcal{S}$ = "generalized monounary algebras".)
2. $\mathcal{A}$ is a type-**2** affine variety. (Much is known. $\mathcal{A}$ = "generalized modules over a finite ring".)
3. $\mathcal{D}$ is a type-**3** discriminator variety. (Finitely generated case is fully known. $\mathcal{S}$ = "generalized Boolean algebras".)

Moreover, it is impossible to interpret the class of graphs into any of $\mathcal{S}, \mathcal{A}, \mathcal{D}$.