MATH 8174: Assignment 11

1. Let $K \trianglelefteq L$. Show that L is solvable if and only if K and L/K are both solvable. Show also that this fails with "nilpotent" in place of "solvable".
2. Prove the following as a corollary of Lie's theorem: If L is a finite dimensional solvable Lie algebra over $\mathbb{C}$, then $L^2 = L^{(1)}$ is nilpotent. You may or may not like to try the following:

(a) Recall that $\mathfrak{t}_n(k)$ (respectively, $\mathfrak{u}_n(k)$) is the Lie subalgebra of $\mathfrak{gl}_n(k)$ consisting of all the upper triangular (respectively, strictly upper triangular) matrices, and also that $\mathfrak{u}_n(k)$ is a nilpotent Lie algebra. Show that $\mathfrak{t}_n(k)^{(1)} = \mathfrak{u}_n(k)$ for any field k.

(b) If $\dim_{\mathbb{C}}(L) = n$, show that $L/Z(L)$ is isomorphic to a subalgebra of $\mathfrak{t}_n(\mathbb{C})$.

(c) From (a) and (b), deduce that $(L^{(1)} + Z(L))/Z(L)$ is isomorphic to a subalgebra of $\mathfrak{u}_n(\mathbb{C})$.

(d) Deduce from (c) that $L^{(1)}$ is nilpotent.

3. Let L be a 1-dimensional Lie algebra over $\mathbb{R}$ with basis $\{x\}$. Let V be a 2-dimensional vector space over $\mathbb{R}$ with basis $\{v_1, v_2\}$, on which x acts by $x.v_1 = v_2$ and $x.v_2 = -v_1$. Show that V is a simple L-module.
4. Let L be a 3-dimensional vector space over a field k, with basis $\{x, y, z\}$. Show that the relations $[x, z] = [y, z] = 0$ and $[x, y] = z$ uniquely define a Lie algebra structure on L.
5. Let L be the Lie algebra of Question 4, and assume now that k has prime characteristic p. Let V_p be a p-dimensional vector space over k with basis $\{v_1, v_2, \ldots, v_p\}$.

(a) Show that there is a well-defined action of L on V_p given by the linear extension of the relations

$$x.v_i = v_{i-1}, \quad y.v_i = iv_{i+1}, \quad z.v_i = v_i,$$

for all $1 \leq i \leq p$, where coefficients and subscripts are reduced modulo p.

(b) Suppose that U is a nonzero submodule of V_p, and let $0 \neq u \in U$ be a nonzero vector in U. Write $u = \sum_{i=1}^{p} \lambda_i v_i$, and let λ_j be the first non-zero term in the sequence $\{\lambda_i\}_{i=1}^{p}$. Show that $y^{p-j}u$ is a nonzero multiple of v_p.

(c) Deduce that V_p is simple, and show that Lie's Theorem fails in characteristic p.