MATH 8174: Assignment 4

1. Find the character table of the alternating group A_4, the dihedral group D_8 of order 8, and the quaternion group Q_8 of order 8. You should notice something interesting in the last two cases.
2. Suppose that a finite group G acts as permutations on a finite set S. What are the character values of the corresponding matrix representation V in terms of fixed point sets of elements of G on S? By counting the set of pairs $(g, s) \in G \times S$ such that $gs = s$ in two different ways, show that

$$\sum_{g \in G} \mathrm{Tr}(g, V) = \sum_{s \in S} |\mathrm{Stab}_G(s)| = |G|.(\text{number of orbits of } G \text{ on } S).$$

   Deduce that the number of copies of the trivial representation as a summand of V is equal to the number of orbits of G on S.
3. (Note: this question is **long**!) Denote by $G = GL(3, 2)$ the group of 3×3 invertible matrices over the field $\mathbb{F}_2$ of two elements.

(i) Show that G has six conjugacy classes of elements, and that the orders of the corresponding elements are 1, 2, 3, 4, 7 and 7.

(ii) By considering the permutation action of G on the seven one-dimensional subspaces of a three-dimensional vector space over $\mathbb{F}_2$, show that G has an irreducible six-dimensional complex representation. Find its character.

(iii) By considering the permutation action of G on the eight Sylow 7-subgroups, show that G has an irreducible seven-dimensional complex representation. Find its character.

(iv) Using the exterior square of the irreducible six-dimensional character, find an irreducible eight-dimensional character.

(v) What are the dimensions of the remaining two irreducible representations? Use the orthogonality relations to write down their character values. You may have trouble with the elements of order seven. Remember that their character values must be sums of appropriate seventh roots of unity.

4. Suppose that a finite group G acts transitively on a finite set S, and let V be the corresponding matrix representation. Show that, as G-modules, we have $V^* \cong V$. Deduce using question 2 that the number of orbits of G on $S \times S$ is equal to the norm, $\langle \chi, \chi \rangle$, of the character of V. Show that the number of orbits of G on $S \times S$ is also equal to the number of *suborbits* of G on S, meaning the number of orbits of $\mathrm{Stab}_G(s)$ on S for a typical element $s \in S$.