MATH 6250: Final examination. May 1–5, 2015.

Put **your name** on each answer sheet. Answer all questions.

Justify your answers in full. This examination is open book and you may use any other resources you like, with the exception of other people. Solutions must be submitted before 1pm on Tuesday May 5.

1. Let R be a ring in which every element is idempotent (i.e., $x^2 = x$ for all $x \in R$).

(i) Show that R is commutative and von Neumann regular.

(ii) Give explicit examples to show that R may or may not be semisimple.

2. Let k be a field and let V be a two-dimensional vector space over k.

(i) Show that, over any field k, the multiplicative subgroup $G \leq GL(V)$ generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is isomorphic to the symmetric group S_3. [Hint: this is related to the "fixed points minus one" representation of S_3.]

(ii) In the case $k = \mathbb{C}$, determine (a) whether V is an absolutely irreducible kG-module; (b) whether G is completely reducible; (c) the set of unipotent elements of G and (d) the unipotent radical of G. Your answers to (c) and (d) should be in the form of explicit sets of matrices.

(iii) Repeat (ii) for the case where $k = \mathbb{F}_2$, the field with two elements.

(iv) Repeat (ii) for the case where $k = \mathbb{F}_3$, the field with three elements.

(v) Do any of the cases (ii), (iii) or (iv) have the property that kG is not semisimple but $S = \mathrm{Span}_k(G)$ is semisimple? Explain why or why not.

3. Let $k = \mathbb{F}_3$ be the field with three elements, let G be the group $\mathbb{Z}/4\mathbb{Z}$ and let $R = kG$ be the corresponding group algebra.

(i) Show that, over k, R has a two-dimensional irreducible module M.
Hint: let a generator of G act by the endomorphism $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) Identify the division ring $\mathrm{End}_R(M)$. Hence, or otherwise, express R as a direct product of matrix rings $M_{n_i}(D_i)$ over division rings D_i. (You need to give the numbers n_i and the division rings D_i explicitly.)

(iii) Find an extension $K \geq k$ of k that is a splitting field for G. Describe all the irreducible representations of KG as explicitly as possible, in terms of the structure of K.

4. Let T be the unique nonabelian group of order 12 that has cyclic Sylow subgroups. A presentation for T is given by

$$T = \langle a, b \mid a^4 = b^3 = 1,\ aba^{-1} = b^{-1} \rangle.$$

You may accept the above claims without proof.

(i) Find the character table of T.
[You may find the following facts helpful, but if you use any of them, you must prove them. The group T has six conjugacy classes, four of which contain elements of orders 1, 2, 3 and 6, and the other two of which contain elements of order 4. The commutator subgroup T' has index 4. The center of T has order 2, and $T/Z(T)$ is isomorphic to S_3.]

(ii) Find a field K of characteristic 2 that is a splitting field for T. Describe all the irreducible representations of T over K, and find the dimension of the Jacobson radical, $J(KT)$. [Hint: use the result of Question 2, part (iii).]

(iii) Repeat part (ii) in the case of characteristic 3. [Hint: use the result of Question 3, part (iii).]