1. Definitions

There are more than a few. I’ll use \( \mathbb{R} \) for \( \mathbb{R} \cup \{ \pm \infty \} \).

**Partition:** A *partition* of \([a, b]\) is a collection of points \( \{x_0, \ldots, x_m\} \) so that \( x_0 = a, x_m = b \) and \( x_{i-1} < x_i \) for \( i = 1, \ldots, m \).

**Variation:** The *variation* of \( f \) over \([a, b]\) is defined by

\[
V = V[f; a, b] = \sup_{\Gamma} S_{\Gamma},
\]

where \( S_{\Gamma} \) is the sum

\[
\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|
\]

and the supremum is taken over all partitions \( \Gamma \) of \([a, b]\). Intuitively, the variation is how much \( f \) moves up and down over the interval \([a, b]\).

**Function of Bounded Variation:** one with \( V[f; a, b] \) finite.

**Function of Unbounded Variation:** one with \( V[f; a, b] = +\infty \).

**Positive Variation:** The *positive variation* of \( f \) over \([a, b]\) is defined by

\[
P = P[f; a, b] = \sup_{\Gamma} P_{\Gamma},
\]

where

\[
P_{\Gamma} = \sum_{i=1}^{m} (f(x_i) - f(x_{i-1}))^+
\]

and the supremum is taken over all partitions \( \Gamma \) of \([a, b]\). Intuitively, the positive variation is how much \( f \) moves up over the interval \([a, b]\).

**Negative Variation:** The *negative variation* of \( f \) over \([a, b]\) is defined by

\[
N = N[f; a, b] = \sup_{\Gamma} N_{\Gamma},
\]

where

\[
N_{\Gamma} = \sum_{i=1}^{m} (f(x_i) - f(x_{i-1}))^-
\]

and the supremum is taken over all partitions \( \Gamma \) of \([a, b]\). Intuitively, the negative variation is how much \( f \) moves down over the interval \([a, b]\).

**Rectifiable Curve:** Intuitively, a curve with finite length. Formally, the length \( L \) of a curve \( C \) with coordinate functions \( \phi : [a, b] \to \mathbb{R} \) and \( \psi : [a, b] \to \mathbb{R} \) is defined as the supremum (over \( \Gamma \)) of the sums

\[
L(\Gamma) = \sum_{i=1}^{m} \sqrt{(\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1}))^2}
\]
where $\Gamma$ is a partition $\{t_0 = a, \ldots, t_m = b\}$ of $[a, b]$. We say $C$ is a rectifiable curve if $L$ is finite.

**Norm of a Partition:** The norm $|\Gamma|$ of a partition $\Gamma$ is the length of the largest interval of $\Gamma$. That is, if $\Gamma = \{x_0, \ldots, x_n\}$, $|\Gamma| = \max_i \{x_i - x_{i-1}\}$.

**Riemann-Stieltjes Integral:** Intuitively, the Riemann integral with a change of variables built in. Formally, let $f$ and $\phi$ be two functions which are defined and finite on a finite interval $[a, b]$. If $\Gamma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of $[a, b]$, we arbitrarily select intermediate points $\{\xi_i\}_{i=1}^m$ satisfying $x_{i-1} \leq \xi_i \leq x_i$, and write

$$R_\Gamma = \sum_{i=1}^m f(\xi_i)(\phi(x_i) - \phi(x_{i-1})).$$

$R_\Gamma$ is called a Riemann-Stieltjes sum for $\Gamma$, and of course depends on $\xi_i$, $f$, $\phi$, etc, but we don’t bother to indicate this dependence in our notation. Then, if $I = \lim_{|\Gamma| \to 0} R_\Gamma$ exists and is finite, that is, if given $\epsilon > 0$ there is a $\delta > 0$ such that $|I - R_\Gamma| < \epsilon$ for any $\Gamma$ satisfying $|\Gamma| < \delta$, then $I$ is called the Riemann-Stieltjes integral of $f$ with respect to $\phi$ on $[a, b]$, and denoted

$$I = \int_a^b f(x) \, d\phi(x) = \int_a^b f \, d\phi.$$

**Step Function:** A function $\phi$ whose domain may be partitioned into finitely many intervals so that $\phi$ is constant on each interval.

**Lebesgue Outer Measure:** Intuitively, the smallest volume of intervals that cover $E$. Formally, let $E$ be a subset of $\mathbb{R}^n$. Cover $E$ by a countable collection $S$ of $n$-dimensional closed intervals $I_k$, and let

$$\sigma(S) = \sum_{I_k \in S} v(I_k),$$

where $v$ is the $n$-volume of the interval $I_k$. The Lebesgue outer measure (or exterior measure) of $E$, denoted $|E|_e$, is defined by

$$|E|_e = \inf \sigma(S),$$

where the infimum is taken over all such covers $S$ of $E$.

**Lebesgue Measurable Set:** Intuitively, a set which is well approximated by a collection of intervals. Formally, a subset $E$ of $\mathbb{R}^n$ is said to be Lebesgue measurable, or simply measurable, if given $\epsilon > 0$, there exists an open set $G$ such that

$$E \subset G \text{ and } |G - E|_e < \epsilon.$$  

(Note that open sets are precisely those that can be expressed as a countable union of open intervals, hence the intuitive interpretation.)

**Lebesgue Measure:** If $E$ is Lebesgue measurable, we define its measure $|E|$ to be its outer measure $|E|_e$. Intuitively, this is the volume of $E$.

**\(\sigma\)-Algebra:** A collection of sets $\Sigma$ that is closed under complements, countable unions, and countable intersections. (The first two properties imply the third.)

**Borel Set:** A set obtainable by complements, countable unions, and countable intersections from open sets in finitely many steps. Alternatively, a member of the $\sigma$-algebra generated by the open subsets of $\mathbb{R}^n$. 
Almost Everywhere: A property is said to hold *almost everywhere* (or a.e if we’re feeling lazy) on a set $E$, if the set of points of $E$ where it does *not* hold has measure zero.

Measurable Function: A function $f : \mathbb{R}^n \to \mathbb{R}$ so that the preimage of each interval $(a, \infty]$ is a measurable set for each finite $a \in \mathbb{R}$. Intuitively, these are the functions so that the Lebesgue integral makes sense.

Upper-semicontinuous Function: Intuitively, a function whose limsups are not too large. Formally, $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous (or usc if we’re feeling lazy) at $x_0$ if

$$\limsup_{x \to x_0; x \in E} f(x) \leq f(x_0).$$

Alternatively, we have exactly one half of the $\epsilon$-$\delta$ definition of continuity: $f$ is usc at $x_0$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that for all $x \in E$ with $|x - x_0| < \delta$ it follows that

$$f(x) - f(x_0) < \epsilon.$$

Lower-semicontinuous Function: Intuitively, a function whose liminfs are not too small. Formally, $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous (or lsc if we’re feeling lazy) at $x_0$ if

$$\liminf_{x \to x_0; x \in E} f(x) \geq f(x_0).$$

Alternatively, we have the other half of the $\epsilon$-$\delta$ definition of continuity: $f$ is lsc at $x_0$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that for all $x \in E$ with $|x - x_0| < \delta$ it follows that

$$-\epsilon < f(x) - f(x_0).$$

Property $\mathcal{C}$: Intuitively, discontinuous in only a set of arbitrarily small measure. Formally, $f$ has property $\mathcal{C}$ on $E$ if given $\epsilon > 0$, there is a closed set $F \subset E$ such that

1. $|E - F| < \epsilon$
2. $f$ is continuous relative to $F$.

If $E$ is measurable, then this is equivalent to $f$ being measurable on $E$. (This definition appears to be endemic to our textbook.)

Convergence in Measure: A sequence of functions $f_k$ is said to converge in measure to $f$ on $E$ (written $f_k \overset{m}{\to} f$) if for every $\epsilon > 0$,

$$\lim_{k \to \infty} |\{x \in E : |f(x) - f_k(x)| > \epsilon\}| = 0.$$

Intuitively, the size of the set where $f_k$ is far from $f$ can be made to have arbitrarily small measure by choosing $k$ large.

Lebesgue Integral: This is defined in two steps. For the first step, let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be a non-negative function. Define

$$R(f, E) = \{(x, y) \in \mathbb{R}^{n+1} : x \in E, y \in \mathbb{R}, 0 \leq y \leq f(x)\}.$$ 

This is the region between 0 and the graph of $f$. If $R(f, E)$ is measurable, we define the Lebesgue integral of $f$ over $E$ as

$$|R(f, E)|_{n+1} = \int_E f(x) \, dx.$$
In the case that $f$ is not non-negative, we define
\[
\int_E f(x) \, dx = \int_E f^+ \, dx - \int_E f^- \, dx
\]
provided that at least one of the integrals on the right is finite.

**Simple Function:** A function $f$ is simple if it takes finitely many values.

**Integrable Function:** A function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is integrable if $\int_E f$ exists and is finite.

**$L^p$ Space:** $L^p(E)$, $0 < p < \infty$ is the set of functions $f : E \to \mathbb{R}$ so that $|f|^p$ is integrable over $E$. In particular, if $f$ is integrable, $f \in L(E)$.

**Equimeasurable:** Two functions $f, g : E \subset \mathbb{R}^n \to \mathbb{R}$ are equimeasurable or equidistributed if
\[
|\{x \in E : f(x) > \alpha\}| = |\{x \in E : f(x) > \alpha\}|
\]
for all $\alpha$. In the notation of §5.4, $\omega_{f,E} = \omega_{g,E}$.

**Convolution:** If $f$ and $g$ are measurable functions in $\mathbb{R}^n$, their convolution $(f * g)(x)$ is defined by
\[
(f * g)(x) = \int_{\mathbb{R}^n} f(x - t)g(t) \, dt.
\]

**Set Function:** A set function is a real-valued function $F$ defined on a $\sigma$-algebra $\Sigma$ of measurable sets such that
1. $F(E)$ is finite for every $E \in \Sigma$,
2. $F$ is countably additive; i.e., if $E = \bigcup_k E_k$ is a union of disjoint $E_k \in \Sigma$, then
\[
F(E) = \sum_k F(E_k).
\]

**Indefinite Integral:** If $f \in L(A)$, where $A$ is a measurable subset of $\mathbb{R}^n$, the indefinite integral of $f$ is defined to be the set function
\[
F(E) = \int_E f,
\]
where $E$ is any measurable subset of $A$.

**Continuous Set Function:** A set function $F(E)$ is called continuous if $F(E) \to 0$ as the diameter $\sup\{||x - y| : x, y \in E\}$ tends to 0; that is, $F(E)$ is continuous if, given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(E)| < \epsilon$ whenever the diameter of $E$ is less than $\delta$.

**Absolutely Continuous Set Function:** A set function $F(E)$ is called absolutely continuous if $F(E)$ tends to zero as the measure of $E$ tends to zero. If you like $\epsilon$s and $\delta$s, $F$ is absolutely continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(E)| < \epsilon$ whenever the measure of $E$ is less than $\delta$.

**Hardy-Littlewood Maximal Function:** If $f$ is a function defined on $\mathbb{R}^n$ and integrable over every cube $Q$, we define the Hardy-Littlewood maximal function of $f$ by
\[
f^*(x) = \sup_{|Q|} \frac{1}{|Q|} \int_Q |f(y)| \, dy
\]
where the supremum is taken over all $Q$ with edges parallel to the coordinate axes and center $x$. Other sources tend to define this in terms of balls centered at $x$ rather than cubes.
Weak $L(\mathbb{R}^n)$: A function $f$ belongs to $\text{weak } L(\mathbb{R}^n)$ if there is a constant $c$ independent of $\alpha$ so that
\[ |\{ x \in \mathbb{R}^n : |f(x)| > \alpha \} | \leq \frac{c}{\alpha} \]
for all $\alpha > 0$. These are functions that obey Tchebyshev’s Inequality except they get the constant wrong.

Locally Integrable: A function $f$ is locally integrable on $E$ if it is integrable over every bounded measurable subset of $E$.

Point of Density: $x$ is a point of density of $E$ if
\[ \lim_{Q \searrow x} \frac{|E \cap Q|}{|Q|} = 1. \]

Point of Dispersion: $x$ is a point of dispersion of $E$ if
\[ \lim_{Q \searrow x} \frac{|E \cap Q|}{|Q|} = 0. \]

Cover in the Sense of Vitali: A family $K$ of cubes is said to cover a set $E$ in the Vitali sense if for every $x \in E$ and $\eta > 0$, there is a cube in $K$ containing $x$ whose diameter is less than $\eta$.

Absolutely Continuous Function: A finite function $f$ on a finite interval $[a,b]$ is said to be absolutely continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that for any collection $\{[a_i, b_i]\}$ (finite or countable) of nonoverlapping subintervals of $[a,b]$,

Singular Function: A function $f$ is singular on $[a,b]$ if $f'$ is zero a.e. in $[a,b]$.

Convex Function: Let $\phi$ be defined and finite on an interval $(a,b)$. We say $\phi$ is convex in $(a,b)$ if for every $[x_1, x_2]$ in $(a,b)$, the graph of $\phi$ on $[x_1, x_2]$ lies on or below the line segment connecting the points $(x_1, \phi(x_2))$ and $(x_2, \phi(x_2))$. In other words, the region above the graph of $\phi$ is convex.

2. Frequently Cited and Otherwise Important Theorems

3. Examples

3.1. The Dirichlet Function. This is the function defined on $[0,1]$ by
\[ f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \]
This function is integrable, but not Riemann-integrable. It is defined on a finite interval, but has unbounded variation.

3.2. The Cantor Set. You know what this is. It is uncountable, but has measure 0.

3.3. The Cantor-Lebesgue Function. Singular, but not constant. Has bounded variation.