1 The Principle of Mathematical Induction

There are many methods of proving statements in mathematics. Some examples are

- Directly proving a specific case. ($f(x) = |x|$ is not differentiable).
- Letting a quantity be arbitrary, and show the result holds. ($\frac{d}{dx}[x^n] = nx^{n-1}$).
- Proving a fact in a general setting (Every differentiable function is continuous).

The principle of Mathematical Induction, or just induction for short, is a general proof method similar to above, but is quite specific in its usage.

1.1 Definition of Induction

A proof by Induction holds because of a property of the natural numbers:

**Axiom.** Let $p(n)$ be a statement depending on $n \in \mathbb{N} = \{1, 2, 3, 4, \ldots \}$ such that

1. $p(1)$ is true
2. for all $n \in \mathbb{N}, p(n)$ is true $\Rightarrow p(n+1)$ is true.

Then $p(n)$ is true for all $n \in \mathbb{N}$.

The best way to think about this is to think about dominos. Imagine a long line of dominos, going off to infinity. Property 2 tells us that if a domino gets knocked over then the next one will also get knocked over. Property 1 tells us we can knock over the first domino. If both of these properties are true, we can then knock over the first domino, which knocks over the second, which in turn knocks over the third, which then knocks over the forth, which then $\ldots$ As you can see, if we can show these two properties hold for a statement $p$, then the statement must always be true.

1.2 Examples

Although thinking of this method of proof gives a nice picture of why it works, using it can be kind of strange. Namely, you show a statement is true for a (usually) simple number. From there, out of the woodwork you assume the statement is true for a number, and you use that fact to show that the statement is true for the next number. For example, consider the statement

$p(n) : 1 + 3 + 5 + 7 + 9 + \cdots + 2n - 1 = n^2$.

We would like to show that this statement holds for all natural numbers $n$.

**Proof.**

1. $p(1) : 1 = 1^2$ is true.

2. Suppose $p(n)$ is true.

   Then $1 + 3 + 5 + \cdots + 2n - 1 = n^2$

So, $1 + 3 + 5 + \cdots + 2(n+1) - 1 - 1 = 1 + 3 + 5 + \cdots + 2n - 1 + 2(n+1) - 1$

   $= n^2 + 2(n+1) - 1$

   $= n^2 + 2n + 1$

   $= (n+1)^2$. 
Thus, \( p(n + 1) : 1 + 3 + 5 + \cdots + 2(n + 1) - 1 = (n + 1)^2 \) is true.
Hence, \( p(n) \Rightarrow p(n + 1) \) for all \( n \).
Therefore, by the Principle of Mathematical Induction, \( p(n) \) is true for all \( n \in \mathbb{N} \).

Notice that most of the work involves assuming that the statement holds for an arbitrary \( n \), and then showing that the next is true. Another example:

**Claim:** \( n + 3 < 5n^2 \) for all \( n \in \mathbb{N} \).

**Proof.**

1. \( p(1) : 1 + 3 < 5(1)^2 \) is true.

2. Suppose \( p(n) \) is true.
   Then \( n + 3 < 5n^2 \).
   So, \( 5(n + 1)^2 = 5n^2 + 10n + 1 > (n + 3) + 0 + 1 = (n + 1) + 3 \). Thus, \( p(n + 1) \) is true.
   Hence, \( p(n) \Rightarrow p(n + 1) \) for all \( n \).
   Therefore, by the Principle of Mathematical Induction, \( p(n) \) is true for all \( n \in \mathbb{N} \).

One last example:

**Claim:** \( 4^n - 1 \) is divisible by 3 for all \( n \in \mathbb{N} \).

**Proof.**

1. \( p(1) : "4^1 - 1 = 3" \) is divisible by 3 is true.

2. Suppose \( p(n) \) is true.
   Then 3 divides \( 4^n - 1 \).
   So \( 4^n - 1 = 3k \) for some integer \( k \).
   Note that \( 4^{n+1} - 1 = 4 \cdot 4^n - 1 = 3 \cdot 4^n + (4^n - 1) = 3 \cdot 4^n + 3k = 3(4^n + k) \).
   So, \( 4^{n+1} - 1 \) is divisible by 3.
   Thus, \( p(n + 1) \) is true.
   Hence, \( p(n) \Rightarrow p(n + 1) \) for all \( n \).
   Therefore, by the Principle of Mathematical Induction, \( p(n) \) is true for all \( n \in \mathbb{N} \).

### 1.3 Problems

Show that each of the following hold for all \( n \in \mathbb{N} \).

1. \( 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} \).
2. \( 4^{n+4} > (n + 4)^4 \).
3. \( \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1} \).
4. For differentiable functions \( f_1, f_2, \ldots, f_n \) we have
   \[
   \frac{d}{dx} [f_1 + f_2 + \cdots + f_n] = \frac{d}{dx} [f_1] + \frac{d}{dx} [f_2] + \cdots + \frac{d}{dx} [f_n].
   \]