(19) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.

(a) \( g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix} \)

(b) \( h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix} \)

(20) [1, Section 1.8, Ex 24] An affine transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) has the form \( T(x) = Ax + b \) with \( A \) an \( m \times n \)-matrix and \( b \in \mathbb{R}^m \). Show that \( T \) is not a linear transformation if \( b \neq 0 \).

(21) Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be a linear map such that

\[
T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}
\]

(a) Use the linearity of \( T \) to find \( T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \) and \( T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \).

(b) Determine \( T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \) for arbitrary \( x, y \in \mathbb{R} \).

(22) Give the standard matrices for the following linear transformations:

(a) \( T : \mathbb{R}^2 \to \mathbb{R}^3, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x \\ -x + y \end{bmatrix} \)

(b) the function \( S \) on \( \mathbb{R}^2 \) that scales all vectors to half their length.

(23) Give the standard matrix for the linear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) that rotates points (about the origin) by 60° counterclockwise and then reflects them on the \( x \)-axis.

(24) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the reflection at the line \( 2x + 3y = 0 \). Note that \( T \) is linear.

(a) What is the reflection of the normal vector \( a = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) of the line? What is the reflection of the vector \( b = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \), which is on this line? Make a drawing if necessary.

(b) Write the unit vectors \( e_1, e_2 \) as linear combinations of \( a \) and \( b \).

(c) Use the linearity of \( T \) to find the reflection of the unit vectors \( T(e_1), T(e_2) \) from \( T(a), T(b) \).

(d) Give the standard matrix for \( T \).

(25) Is \( T : \mathbb{R}^3 \to \mathbb{R}^2, x \mapsto \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot x \) injective, surjective, bijective?

(26) [1, cf. Section 1.9, Ex 23/24] True or False? Correct the false statements to make them true.
(a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.

(b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.

(c) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $\in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.

(d) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.

(27) Compute if possible

$$A + 3B, B \cdot A, A \cdot B, A \cdot C, C \cdot A$$

for the matrices

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 1 & 3 \end{bmatrix}.$$

If an expression is undefined, explain why.

**References**