(64) [1, Section 4.2] Let $U, V, W$ be vector spaces, and let $f: U \rightarrow V$ and $g: V \rightarrow W$ be linear mappings.

(a) Show that the composition mapping $h: U \rightarrow W, u \mapsto g(f(u))$ is linear.

**Solution:**
Let $u, v \in U$ and $r \in \mathbb{R}$. Then

$$h(u + v) = g(f(u + v))$$
$$= g(f(u) + f(v)) \quad \text{by linearity of } f,$$
$$= g(f(u)) + g(f(v)) \quad \text{by linearity of } g,$$
$$= h(u) + h(v).$$

Also,

$$h(ru) = g(f(ru))$$
$$= g(rf(u)) \quad \text{by linearity of } f,$$
$$= rg(f(u)) \quad \text{by linearity of } g,$$
$$= rh(u).$$

Thus $h$ is linear.

(b) Does it make sense to ask whether $k: V \rightarrow V, u \mapsto f(g(v))$ is linear?

**Solution:**
No. The vector $g(v)$ is in $W$, but we do not know if $g(v)$ is in $U$. Thus $g(v)$ cannot be an input of $f$.

(65) [1, Section 4.2] Let $U, V$ be vector spaces and $T: U \rightarrow V$ be a linear mapping. Show that $T(0) = 0$.

**Hint:** Write down $T(0 + 0)$ in two different ways.

**Solution:**
We have $T(0) = T(0 + 0) = T(0) + T(0)$. So

$$T(0) = T(0) + T(0).$$

By subtracting $T(0)$ from each side we obtain $0 = T(0)$.

(66) [1, Section 4.2] Let $U, V$ be vector spaces and $T: U \rightarrow V$ be a linear mapping. Show that the range $\text{Rg} T$ is a subspace of $V$.

**Solution:**
We show the subspace conditions. (1) By problem (65) $T(0) = 0$. Thus the zero vector is in the range.

(2) and (3): Let $v_1, v_2 \in \text{Rg} T$ and $r \in \mathbb{R}$. We have to show that $v_1 + v_2 \in \text{Rg} T$ and $rv_1 \in \text{Rg} T$. Since $v_1, v_2$ are in the range, there are $u_1, u_2 \in U$ such that $T(u_1) = v_1$, $T(u_2) = v_2$. Then

$$T(u_1 + u_2) = T(u_1) + T(u_2) = v_1 + v_2,$$
$$T(ru_1) = rT(u_1) = rv_1.$$
and \( T(\mathbf{u}_2) = \mathbf{v}_2 \). Now
\[
\mathbf{v}_1 + \mathbf{v}_2 = T(\mathbf{u}_1) + T(\mathbf{u}_2) = T(\mathbf{u}_1 + \mathbf{u}_2).
\]
This means \( \mathbf{v}_1 + \mathbf{v}_2 \) is the image of \( \mathbf{u}_1 + \mathbf{u}_2 \). Thus \( \mathbf{v}_1 + \mathbf{v}_2 \in \text{Rg} \, T \).

Also,
\[
r\mathbf{v}_1 = rT(\mathbf{u}_1) = T(r\mathbf{u}_1).
\]
Thus \( r\mathbf{v}_1 \) is the image of \( r\mathbf{u}_1 \), and hence \( r\mathbf{v}_1 \in \text{Rg} \, T \).

(67) [1, Section 4.3] Let \( V = \{ f: \mathbb{R} \to \mathbb{R} \} \) be a vector space of functions. Is the set \( \{ \cos t, \sin t, \sin(t + \frac{\pi}{4}) \} \) linearly independent?

Hint: Use the formula for \( \sin(\alpha + \beta) \).

**Solution:**
No. Since \( \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \), we have
\[
\sin(t + \frac{\pi}{4}) = \sin(t) \cos(\frac{\pi}{4}) + \cos(t) \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \sin(t) + \frac{1}{\sqrt{2}} \cos(t).
\]
Thus \( \sin(t + \frac{\pi}{4}) \) is a linear combination of \( \sin(t) \) and \( \cos(t) \). The vectors are not linearly independent.

(68) [1, Section 4.4] Let \( B = ( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} ) \) and \( C = ( \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} ) \) be bases of \( \mathbb{R}^2 \).

(a) Find the standard matrix for \( f: \mathbb{R}^2 \to \mathbb{R}^2, [\mathbf{u}]_B \mapsto \mathbf{u} \).

**Solution:**
Since \( \mathbf{u} = P_B[\mathbf{u}]_B \), the standard matrix of \( f \) is \( P_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \).

(b) Find the standard matrix for \( g: \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{u} \mapsto [\mathbf{u}]_C \).

**Solution:**
Since \( [\mathbf{u}]_C = P_C^{-1}\mathbf{u} \), the standard matrix of \( g \) is \( P_C^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \).

(c) Find the standard matrix for \( h: \mathbb{R}^2 \to \mathbb{R}^2, [\mathbf{u}]_B \mapsto [\mathbf{u}]_C \). Hint: \( h(\mathbf{x}) = g(f(\mathbf{x})) \).

**Solution:**
Since
\[
h(\mathbf{x}) = g(f(\mathbf{x})) = g(P_B\mathbf{x}) = P_C^{-1}P_B\mathbf{x},
\]
the standard matrix of \( h \) is \( P_C^{-1}P_B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix} \).

(69) [1, Sections 4.1–4.4] Let \( B = (t, 2 + t, t^2) \) and \( C = (1, t + t^2, t^3) \) be bases of \( \mathbb{P}_2 \). Let \( h: \mathbb{P}_2 \to \mathbb{P}_2 \) be the linear mapping \( h(\mathbf{e}_1), h(\mathbf{e}_2), h(\mathbf{e}_3) \). Hint: If \( [\mathbf{u}]_B = \mathbf{e}_1 \), then \( [\mathbf{u}]_C = h(\mathbf{e}_1) = ? \)

(b) Give the standard matrix of \( h \).

(c) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{P}_2 \) such that \( [\mathbf{v}_1]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, [\mathbf{v}_2]_B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \text{and} \ [\mathbf{v}_3]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \). Find \( [\mathbf{v}_1]_C, [\mathbf{v}_2]_C, [\mathbf{v}_3]_C \).
Solution:
If \([u]_B = e_1\), then \(u = 1t + 0(2 + t) + 0t^2 = t\). In order to compute \([u]_C\), we have to find \(x_1, x_2, x_3 \in \mathbb{R}\) such that \(x_1 + x_2(t + t^2) + x_3t^2 = u = t\). I.e.,
\[
x_1 + x_2t + (x_2 + x_3)t^2 = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2.
\]
\(\square\)

(a) (2 points) We compare the coefficients of the LHS and RHS and obtain
\[
x_1 = 0, \ x_2 = 1, \ x_2 + x_3 = 0.
\]
This yields
\[
h(e_1) = [u]_C = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
\]

By similar computations, we obtain the following:
If \([u]_B = e_2\), then \(u = 2 + t\) and \(h(e_2) = [u]_C = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\).
If \([u]_B = e_3\), then \(u = t^2\) and \(h(e_3) = [u]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\).

(b) (1 point) The standard matrix of \(h\) is given by
\[
A = [h(e_1) \ h(e_2) \ h(e_3)] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.
\]

(c) (2 points)
\[
[v_1]_C = h([v_1]_B) = A[v_1]_B = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix},
\]
\[
[v_2]_C = h([v_2]_B) = A[v_2]_B = A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix},
\]
\[
[v_3]_C = h([v_3]_B) = A[v_3]_B = A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.
\]

(70) [1, Section 4.3] Let \(b_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \ b_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}\).

(a) Find vectors \(u_1, \ldots, u_k\) such that \((b_1, b_2, u_1, \ldots, u_k)\) is a basis for \(\mathbb{R}^3\).
(b) Find vectors \(v_1, \ldots, v_\ell\) such that \((b_3, v_1, \ldots, v_\ell)\) is a basis for \(\mathbb{R}^3\).
Prove that your choices for (a) and (b) form a basis.

Solution:
Both bases have 3 vectors. Thus \(k = 1\) and \(\ell = 2\).
(a) One possible choice is $u_1 = e_1$. We show that $(b_1, b_2, e_1)$ is a basis by reducing the following augmented matrix to echelon form:

$$
\begin{bmatrix}
1 & 1 & 1 & 0 \\
2 & 1 & 0 & 0 \\
-1 & 3 & 0 & 0
\end{bmatrix} \sim \cdots \sim 
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

(b) One possible choice is $v_1 = e_1$ and $v_2 = e_2$. We show that $(b_3, e_1, e_2)$ is a basis by reducing the following augmented matrix to echelon form:

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
2.5 & 0 & 1 & 0 \\
-5 & 0 & 0 & 0
\end{bmatrix} \sim \cdots \sim 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

(71) [1, Sections 4.3, 4.5] Let

$$
A = 
\begin{bmatrix}
-5 & 3 & 11 \\
8 & -5 & -19 \\
0 & 1 & 7 \\
-17 & 5 & -3
\end{bmatrix}
$$

Find bases and dimensions for $\text{Nul } A$, $\text{Col } A$, and $\text{Row } A$, respectively.

**Solution:**

(1 point) We reduce $A$ to reduced echelon form:

$$
A \sim \cdots \sim 
\begin{bmatrix}
1 & 0 & 0 & 5 & 2 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(2 points) For $\text{Nul } A$, we solve $Ax = 0$ and obtain

$$
\text{Nul } A = \{r \begin{bmatrix}
-5 \\
1 \\
0
\end{bmatrix} + s \begin{bmatrix}
-2 \\
-1 \\
0
\end{bmatrix} \mid r, s \in \mathbb{R}\}.
$$

The two vectors form a basis for $\text{Nul } A$.

(1 point) The first three columns of $A$ contain a pivot. Thus they form a basis

$$
B = \left( 
\begin{bmatrix}
-5 \\
3 \\
11
\end{bmatrix}, \begin{bmatrix}
8 \\
-5 \\
-19
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
7
\end{bmatrix} \right)
$$

for $\text{Col } A$. 

(1 point) The nonzero rows in any echelon form of $A$ form a basis. E.g.,

$$C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
5 & 1 & -5 \\
2 & 1 & 0
\end{pmatrix}$$

is a basis for Row $A$. □

(72) [1, Section 4.5] A $177 \times 35$ matrix $A$ has 19 pivots. Find $\dim \text{Nul} A$, $\dim \text{Col} A$, $\dim \text{Row} A$, and $\text{rank} A$.

**Solution:**

The number of pivots, $\dim \text{Row} A$, $\dim \text{Col} A$, and the rank are equal. So

$$\dim \text{Row} A = \dim \text{Col} A = \text{rank} A = 19.$$ 

By the rank theorem, $\dim \text{Nul} A + \text{rank} A = 35$. Thus

$$\dim \text{Nul} A = 35 - 19 = 16.$$ 

Thus □

**References**