1 \textbf{Sequences and Series}

1. Use $a_n = \frac{2}{n+1}$ for the following questions.

   (a) Write the sequence made up of the given terms. Calculate the first 3 terms of the sequence.
   
   Sequence: $\frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \ldots$

   (b) Write the series made up of the given terms. Calculate the first 3 partial sums.
   
   Series: $\frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \ldots$
   
   $S_1 = 1$ \quad $S_2 = 1 + \frac{2}{3}$ \quad $S_3 = 1 + \frac{2}{3} + \frac{1}{2}$

2. Does the sequence you wrote above converge? If so, to what?

   Yes, the sequence converges to 0, since
   
   $\lim_{n \to 0} \frac{2}{n+1} = 0$

3. Does the series you wrote above converge?

   The series does not converge

4. Describe the difference between a sequence and a series.

   A sequence is a list of terms, and a series is all of the terms in a sequence added together.
2 Geometric Series

1. Which of the following are geometric series? How can you tell?

   \( \sqrt{a} \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^n \)
   \( \sqrt{b} \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^{2n} \)
   \( \sqrt{c} \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^{n+1} \)

   \[ \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n \rightarrow \frac{a_2}{a_1} \neq \frac{a_3}{a_2} \] 
   \( \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n \rightarrow \frac{3}{6} = \frac{3}{3} = 1 \neq \frac{3}{9} \)
   \( \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n \rightarrow \frac{-16}{4} \neq \frac{-4}{9} \)

2. \( \sum_{n=0}^{\infty} 7\left(\frac{2}{3}\right)^n \) is a geometric series.

   (a) Write down (expand) the first few partial sums of the given series.
   \[ S_1 = 7 \quad S_2 = 7 + 7\left(\frac{2}{3}\right) \quad S_3 = 7 + 7\left(\frac{2}{3}\right) + 7\left(\frac{2}{3}\right)^2 \]
   \[ = \frac{7 - 7\left(\frac{2}{3}\right)^n}{1 - \left(\frac{2}{3}\right)} \]

   (b) What is the \( n^{th} \) partial sum of the given series?
   \[ S_n = \frac{7 - 7\left(\frac{2}{3}\right)^n}{1 - \left(\frac{2}{3}\right)} \]
   \[ \text{note that in } S_3, \text{ the exponent of } \left(\frac{2}{3}\right) \text{ in the numerator was, which matches our formula here} \]

   (c) What does the given series converge to?
   \[ a = 7, \quad r = \frac{2}{3}, \quad \frac{a}{1-r} = \frac{7}{1 - \frac{2}{3}} = \frac{7}{\frac{1}{3}} = 21 \]

3. \( \sum_{n=0}^{\infty} 7\left(\frac{2}{3}\right)^{2n} \) is a geometric series.

   (a) Write down (expand) the first few partial sums of the given series.
   \[ S_1 = 7\left(\frac{2}{3}\right)^0 \quad S_2 = 7\left(\frac{2}{3}\right)^0 + 7\left(\frac{2}{3}\right)^2 \quad S_3 = 7\left(\frac{2}{3}\right)^0 + 7\left(\frac{2}{3}\right)^2 + 7\left(\frac{2}{3}\right)^4 \]
   \[ = \frac{7\left(\frac{2}{3}\right)^6 - 7\left(\frac{2}{3}\right)^{10}}{1 - \left(\frac{2}{3}\right)^2} \]

   (b) What is the \( n^{th} \) partial sum of the given series?
   \[ S_n = \frac{7\left(\frac{2}{3}\right)^6 - 7\left(\frac{2}{3}\right)^{n+3}}{1 - \left(\frac{2}{3}\right)^2} \]
   \[ \text{note that when } n = 2, 3, \text{ this formula gives the correct exponent for } \left(\frac{2}{3}\right)^{n+3} \text{ in the numerator} \]

   (c) What does the given series converge to?
   \[ a = 7\left(\frac{2}{3}\right)^6, \quad r = \left(\frac{2}{3}\right)^2, \quad \frac{a}{1-r} = \frac{7\left(\frac{2}{3}\right)^6}{1 - \left(\frac{2}{3}\right)^2} \]
3 Integral Comparison

If possible, use the $n^{th}$ term (divergence) test, the integral comparison test, or the $p$-series test to determine whether the following series converge or diverge. State which test you used, and if none of them apply, explain why.

1. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
   
   Using $u = \ln(x)$, $\int \frac{\ln(x)}{x} \, dx = \frac{\ln(x)^2}{2} + c$
   
   $\int_{b}^{\infty} \frac{\ln(x)^2}{2} \, dx = \lim_{b \to \infty} \frac{\ln(b)^2}{2}$ diverges, so the series diverges.

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{2}}$
   
   Using $u = \ln(x)$, $\int \frac{1}{x(\ln(x))^{2}} \, dx = -\frac{1}{\ln(x)} + c$
   
   $\int_{b}^{\infty} \frac{1}{x(\ln(x))^{2}} \, dx = \lim_{b \to \infty} \frac{-1}{\ln(b)} + \frac{1}{\ln(2)}$
   
   The integral converges, so the series converges as well.

3. $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$
   
   $\lim_{n \to \infty} \frac{n}{\ln(n)} \neq 0$, so the series diverges by the $n^{th}$ term test.

4. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, with $p = 2$, $p > 1$, so the series converges by the $p$-test.

5. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
   
   $\lim_{n \to \infty} \frac{\sin(n)}{n^2} = 0$, so the $n^{th}$ term test doesn't apply.

6. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, with $p = 2$, $p > 1$, so the series converges by the $p$-test.

7. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, with $p = 2$, $p > 1$, so the series converges by the $p$-test.

8. $\sum_{n=1}^{\infty} \frac{1}{n}$
   
   $\lim_{n \to \infty} \frac{1}{n} \neq 0$, so the series diverges by the $n^{th}$ term test.
4 Comparison Tests

For each of the following series, try to determine if the series converges or diverges. For practice, try both the term-size comparison test and the limit comparison test to see if one or both or neither works, explaining why.

1. \[ \sum_{n=1}^{\infty} \frac{1}{n + \ln(n)} \] limit comparison to \( \frac{1}{n} \) tells us that since \( \sum \frac{1}{n} \) diverges, \( \frac{1}{n + \ln(n)} \) diverges too
   term comparison to \( \frac{1}{n} \) tells us nothing, since the inequality goes the wrong way. Term comparison to \( \frac{1}{n} \) tells us our series diverges

2. \[ \sum_{n=1}^{\infty} \frac{2}{n \ln(n)^{3/2}} \] There is no obvious comparison (term or limit) to help us here. This is a case for integral comparison!

3. \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \] limit comparison:
   \[ \lim_{n \to \infty} \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3 - n^2}}{n^2} = 1 \] so since \( \sum \frac{1}{n^2} \) converges, the series converges
   \[ \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \] diverges, the series diverges

4. \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \] limit comparison:
   \[ \lim_{n \to \infty} \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3 - n^2}}{n^2} = 1 \] so since \( \sum \frac{1}{n^2} \) converges, the series converges
   \[ \frac{n^2 + 1}{\sqrt{n^3 - n^2}} \] diverges, the series diverges

5. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 - n} \] Term comparison tells us the series diverges
   Limit comparison:
   \[ \lim_{n \to \infty} \frac{1}{n^2 - n} \cdot \frac{n^2 - n}{n} = 1 \] so since \( \sum \frac{1}{n^2} \) converges, the series converges

6. \[ \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} \] Intuitively, we expect \( \sin^2(n) \) to behave like \( \frac{1}{n^2} \). However, term comparison requires \( 0 \leq a_n \leq b_n \), so it does not apply. \( \lim \sin(n) \cdot n^2 = \lim \sin(n) \) which diverges, so the limit comparison doesn't help either.

7. \[ \sum_{n=1}^{\infty} e^{-(n+1)} \] Both limit comparison and term comparison to \( e^{-n} \) tell us the series converges. This could also be shown using integral comparison

8. \[ \sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n} \] since \( \frac{1}{n} \) diverges, the series diverges

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