Worksheet Purpose: A few weeks ago we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge (where all integrands are positive). Similarly a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge (where all integrands are positive). In problems 1-7 you’ll apply a similar strategy to determine if certain series converge or diverge. Additionally, in problems 8 and 9 you’ll apply a different method (using limits) to determine if a series converges or diverges.

1. For each of the following situations, determine if \( \sum_{n=1}^{\infty} c_n \) converges, diverges, or if one cannot tell without more information.
   (a) \( 0 \leq c_n \leq \frac{1}{n} \) for all \( n \), we can conclude that \( \sum c_n \) \underline{converges}.
   (b) \( \frac{1}{n} \leq c_n \) for all \( n \), we can conclude that \( \sum c_n \) \underline{diverges}.
   (c) \( 0 \leq c_n \leq \frac{1}{n^2} \) for all \( n \), we can conclude that \( \sum c_n \) \underline{converges}.
   (d) \( \frac{1}{n^2} \leq c_n \) for all \( n \), we can conclude that \( \sum c_n \) \underline{unknown, not enough info.}.
   (e) \( \frac{1}{n^2} \leq c_n \leq \frac{1}{n} \) for all \( n \), we can conclude that \( \sum c_n \) \underline{unknown, not enough info.}.

2. Follow-up to problem 1: For each of the cases above where you needed more information, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given conditions.

3. Fill in the blanks:

<table>
<thead>
<tr>
<th>The Comparison Test (also known as Term-size Comparison Test or Direct Comparison Test)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose that ( \sum a_n ) and ( \sum b_n ) are series with positive terms.</td>
</tr>
<tr>
<td>- If ( \sum b_n ) \underline{converges} and ( a_n \leq b_n ), then ( \sum a_n ) \underline{converges}.</td>
</tr>
<tr>
<td>- If ( \sum b_n ) \underline{diverges} and ( a_n \geq b_n ), then ( \sum a_n ) \underline{diverges}.</td>
</tr>
</tbody>
</table>

Note: in the above theorem and for the rest of this worksheet, we will use \( \sum b_n \) to represent the series whose convergence/divergence we already know (p-series or geometric), and \( \sum a_n \) will represent the series we are trying to determine convergence/divergence of.
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Now we’ll practice using the Comparison Test:

4. Let $a_n = \frac{1}{2^n + n}$ and let $b_n = \left(\frac{1}{2}\right)^n$ for $n \geq 1$, both sequences with positive terms.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

(b) How do the size of the terms $a_n$ and $b_n$ compare?

(c) What can you conclude about $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$?

5. Let $a_n = \frac{1}{n^2 + n + 1}$, a sequence with positive terms.
   Consider the rate of growth of the denominator. This hints at a choice of:
   
   $b_n =$ \underline{another positive term sequence}.

(a) Does $\sum b_n$ converge or diverge? Why?

(b) How do the size of the terms $a_n$ and $b_n$ compare?

(c) What can you conclude about $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$?

6. Use the Comparison Test to determine if $\sum_{n=2}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 - 2}$ converges or diverges.
7. Use the Comparison test to determine if \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3 + n}} \) converges or diverges.

8. Disappointingly, sometimes the Comparison Test doesn’t work like we wish it would. For example, let \( a_n = \frac{1}{n^2 - 1} \) and \( b_n = \frac{1}{n^2} \) for \( n \geq 2 \).

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if \( \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \) converges or diverges?

(b) Show that \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \).

(c) Since \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \), we know that \( a_n \approx b_n \) for large values of \( n \). Do you think that \( \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \) must converge?
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When we have chosen a good series to compare to, but the inequalities don’t work in our favor, we use the Limit Comparison Test instead of the Comparison Test.

**The Limit Comparison Test**

Suppose \( a_n > 0 \) and \( b_n > 0 \) for all \( n \). If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \), where \( c \) is finite and \( c > 0 \), then the two series \( \sum a_n \) and \( \sum b_n \) either both converge or both diverge.

Now we’ll practice using the Limit Comparison Test:

9. Determine if the series

\[
\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}
\]

converges or diverges.