Area accumulation functions and the FTC: an analytical perspective

1. Let $F(x) = \int_3^x e^{5t} \, dt$

   (a) Find a formula for $F(x)$ by anti-differentiating and substituting.

   $F(x) = \int_3^x e^{5t} \, dt = \left. \frac{1}{5} e^{5t} \right|_3^x = \frac{1}{5} e^{5x} - \frac{1}{5} e^{15}$.

   (b) Differentiate to find $F'(x)$.

   $F'(x) = e^{5x}$.

   (c) Explain your result.

   Using the Evaluation Theorem (a part of the FTC), we found a formula for $F(x)$ by antidifferentiating, substituting and subtracting. Then when we found the derivative of $F'(x)$ we basically undid our work. So we ended up with the function inside the integral, with the upper limit of integration $x$ substituted for $t$.

   (d) Why does the lower limit of integration not affect the derivative?

   The lower limit is constant and only shifts the area accumulation function; it does not affect its rate of change. Another way to look at it is that when we took the derivative in part (b), that part of the function was a constant so its rate of change was 0.

   (e) Using what you noticed and learned above, find $\frac{d}{dx} \left[ \int_{-5}^x \arctan t \, dt \right]$.

2. Let $F(x) = \int_4^{x^2} \cos t \, dt$

   (a) Find a formula for $F(x)$ by anti-differentiating.

   (b) Differentiate to find $F'(x)$. Look at your answer and notice how it relates to the definition of $F(x)$. 

   In summary:

   The Fundamental theorem of Calculus, Part 1: If $f$ is continuous on $[a, b]$, then

   $$ \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x) \quad \text{(for } a < x < b) \text{.}$$

   Worded differently, if $F(x) = \int_a^x f(t) \, dt$, then $F'(x) = f(x)$. 
(c) Using what you noticed and learned above, find \( \frac{d}{dx} \left[ \int_{2}^{\sin x} \ln t \, dt \right] \).

In summary:

\[
\text{If } F(x) = \int_{0}^{g(x)} f(t) \, dt, \text{ then } F'(x) = \text{______________}. \\
\]

3. If \( F(x) = \int_{x}^{0} f(t) \, dt \), what is \( F'(x) \)? Hint: notice that \( F(x) = -\int_{0}^{x} f(t) \, dt \).

4. If \( F(x) = \int_{3x}^{x^2} \sin t \, dt \), what is \( F'(x) \)? Hint: the integral can be broken into two parts, so

\[
F(x) = \int_{3x}^{0} \sin t \, dt + \int_{0}^{x^2} \sin t \, dt. \\
\]

In summary:

\[
\text{If } F(x) = \int_{a(x)}^{b(x)} f(t) \, dt, \text{ then } F'(x) = \text{________________________} . \\
\]

5. Use the above result to answer the following: if \( F(x) = \int_{x^3}^{1-x} \frac{t+1}{t-1} \, dt \), what is \( F'(x) \)?