q-Partition Algebra Combinatorics

Tom Halverson
Department of Mathematics
Macalester College
Saint Paul, MN 55105
halverson@macalester.edu

Nathaniel Thiem
Department of Mathematics
University of Colorado at Boulder
Boulder, CO 80309
nathaniel.thiem@colorado.edu

June 24, 2008

Abstract

We compute the dimension $d_{n,r}(q) = \dim(\mathcal{IR}_q^r)$ of the defining module $\mathcal{IR}_q^r$ for the q-partition algebra. This module comes from r-iterations of Harish-Chandra restriction and induction on $\mathrm{GL}_n(\mathbb{F}_q)$. This dimension is a polynomial in q that specializes as $d_{n,r}(1) = n^r$ and $d_{n,r}(0) = B(r)$, the rth Bell number. We compute $d_{n,r}(q)$ in two ways. The first is purely combinatorial. We show that $d_{n,r}(q) = \sum_\lambda f^\lambda(q) m_r^\lambda$, where $f^\lambda(q)$ is the q-hook number and m_r^λ is the number of r-vacillating tableaux. Using a Schensted bijection, we write this as a sum over integer sequences which, when q-counted by inverse major index, gives $d_{n,r}(q)$. The second way is algebraic. We find a basis of $\mathcal{IR}_q^r$ that is indexed by n-restricted q-set partitions of $\{1, \ldots, r\}$, and we show that there are $d_{n,r}(q)$ of these.

1 Introduction

If $V = \mathbb{C}^n$ is the permutation representation of the symmetric group S_n, then the r-fold tensor product representation $V^{\otimes r}$ has dimension n^r. When $n \geq 2r$, the centralizer algebra of S_n acting on $V^{\otimes r}$ is the partition algebra $P_r(n) = \mathrm{End}_{S_n}(V^{\otimes r})$. When $n < 2r$ the partition algebra $P_r(n)$ maps surjectively onto $\mathrm{End}_{S_n}(V^{\otimes r})$. As a bimodule for $(S_n, P_r(n))$ we have

$$V^{\otimes r} \cong \bigoplus_\lambda S_n^\lambda \otimes P_r^\lambda, \tag{1.1}$$

where S_n^λ is the irreducible S_n-module labeled by $\lambda \vdash n$ and P_r^λ is the irreducible $P_r(n)$ module labeled by λ. The partitions $\lambda \vdash n$ which appear in this sum are characterized by the fact that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ then $\lambda_2 + \cdots + \lambda_\ell \leq r$.

The dimensions of the irreducible modules in (1.1) are $\dim(S_n^\lambda) = f_n^\lambda$, the number of standard Young tableaux of shape λ (given by the hook formula), and $\dim(P_r^\lambda) = m_r^\lambda$, the number of r-vacillating tableau of shape λ. Computing dimensions on both sides of (1.1) gives the identity

$$n^r = \sum_{\lambda \vdash n} f_n^\lambda m_r^\lambda. \tag{1.2}$$

A combinatorial proof of (1.2) is given in [HL] by defining a Schensted-like insertion/deletion process to find a bijection

$$\{(a_1, \ldots, a_r)\} \longleftrightarrow \{(P, Q)\} \tag{1.3}$$

between integer sequences $(a_1, \ldots, a_r)$ with $a_j \in \{1, \ldots, n\}$ and pairs (P, Q), where P is a standard Young tableau of shape $\lambda \vdash n$ and Q is an r-vacillating tableau of shape $\lambda \vdash n$.

A q-analog of the partition algebra was first defined in unpublished work of T. Halverson and A. Ram (see for example the abstract [HR1]). To make a q-partition algebra, the symmetric group S_n is replaced by the general linear group $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$. The tensor space $V^{\otimes r}$ is replaced by r iterations of Harish-Chandra restriction Resf and induction Indf (see Section 3) as follows

$$\mathcal{IR}_q^r = (\mathrm{Indf}_{G_{n-1}}^{G_n} \mathrm{Resf}_{G_{n-1}}^{G_n})^r(\mathbb{1}),$$

where $\mathbb{1}$ is the trivial G_n-module. The q-partition algebra is the centralizer algebra $Q_r(n, q) = \mathrm{End}_{G_n}(\mathcal{IR}_q^r)$, and as a bimodule for $(G_n, Q_r(n, q))$, we have

$$\mathcal{IR}_q^r \cong \bigoplus_{\lambda} G_n^\lambda \otimes Q_r^\lambda, \tag{1.4}$$

where G_n^λ is the irreducible, unipotent G_n-module labeled by $\lambda \vdash n$ (see for example [Mac, §IV] for a description of this module) and Q_r^λ is the irreducible $Q_r(n, q)$ module labeled by λ. The dimensions of these modules are $\dim(G_n^\lambda) = m_r^\lambda$ and $\dim(G_n^\lambda) = f_n^\lambda(q)$, the well-known q-analog of f_n^λ given by the q-hook formula. Computing dimensions on both sides of (1.4) gives

$$\dim(\mathcal{IR}_q^r) = \sum_{\lambda} f_n^\lambda(q) m_k^\lambda.$$

Until now, the dimension of the module $\mathcal{IR}_q^r$ was not known. In this paper, we show that

$$\dim(\mathcal{IR}_q^r) = d_{n,r}(q) = \sum_{\ell=1}^{n} S(r, \ell)[n][n-1]\cdots[n-\ell+1]. \tag{1.5}$$

Here $S(r, \ell)$ is a Stirling number of the second kind and $[j] = (q^j - 1)/(q - 1)$ is a q-integer.

We prove (1.5) in two ways. The first is purely combinatorial. Using the Schensted bijection (1.3) we write the dimension as a q-weighted sum over sequences $a = (a_1, \ldots, a_r)$, where each sequence is weighted by an analog of the inverse major index $imaj(a)$. This is done in Proposition 2.3. We then compute this sum directly in Corollary 2.2 to get the second equality in

$$\sum_{\lambda} f_n^\lambda(q) m_r^\lambda = \sum_{(a_1,\ldots,a_r)} q^{imaj(a_1,\ldots,a_r)} = \sum_{\ell=1}^{n} S(r, \ell)[n][n-1]\cdots[n-\ell+1] = d_{n,r}(q). \tag{1.6}$$

The q-polynomial $d_{n,r}(q)$ that appears in this formula has the property that $d_{n,r}(1) = n^r$ and $d_{n,r}(0) = B(r)$, the rth Bell number or number of partitions of $\{1, \ldots, r\}$ into subsets. Thus $d_{n,r}(q)$ is a q-analog of both n^r and $B(r)$, and it interpolates between the two as q ranges through $0 \leq q \leq 1$. In Section 3.1, we define (n-restricted) q-set partitions of $\{1, \ldots, r\}$, and we show that $d_{n,r}(q)$ enumerates these objects. In Section 3, we study the module $\mathcal{IR}_q^r$, and we find a basis of $\mathcal{IR}_q^r$ that is indexed by these q-weighted set partitions of $\{1, \ldots, r\}$. Counting this basis then gives our second proof of the dimension formula in (1.5). It should be noted that the combinatorial proof of this dimension was essential in helping us determine the linear independence of this basis for $\mathcal{IR}_q^r$.

A forthcoming paper by T. Halverson, A. Ram, and N. Thiem will study the q-partition algebra $Q_r(n, q)$. The elements of $Q_r(n, q)$ are viewed as endomorphisms of $\mathcal{IR}_q^r$, so it is crucial in that analysis to have the basis that is derived here in Section 3. The ($q = 1$) partition algebra $P_r(n)$ was defined independently by P. Martin [Mar1, Mar2] and V. Jones [Jo]. See also [HR2] for discussion of the partition algebras. The r-vacillating tableaux in this paper are also appear in [CDDSY] in relation to crossing and nesting set partitions.

T. Halverson was partially supported by the National Science Foundation under grant DMS-0100975. This research was completed while the authors were in residence at the Mathematical Sciences Research Institute (MSRI) in Spring 2008 for the program in Combinatorial Representation Theory. We are grateful for the support and the stimulating research environment at MSRI. We thank Arun Ram for numerous useful conversations, and we thank Vic Reiner and Dennis Stanton for a helpful conversation about the distribution of the statistics *inv* and *imaj* used in the proof of Proposition 2.3.

2 Combinatorial Computation of $d_{n,r}(q)$

This section gives a purely combinatorial derivation of the formula for the q-polynomial $d_{n,r}(q)$. In Section 3, we show that $d_{n,r}(q)$ is the dimension of a module $\mathcal{IR}_q^r$ that comes from r-iterations of Harish-Chandra restriction and induction on the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$.

2.1 The Delete/Insert Schensted Algorithm

For $n, r \in \mathbb{Z}_{>0}$, define

$$\{1, \dots, n\}^r = \left\{ (a_1, \dots, a_r) \;\middle|\; a_j \in \{1, \dots, n\} \right\}.$$

This set has cardinality n^r. For a partition $\lambda \vdash n$, a standard tableau of shape λ is a filling of the boxes of the Young diagram of λ with integers $1, \dots, n$ such that the rows increase left-to-right and the columns increase top-to-bottom. As in [HL] we define an algorithm that maps sequences in $\{1, \dots, n\}^r$ to standard tableaux. Let $a = (a_1, a_2 \dots, a_r)$ and recursively define P_i and $P_{i+\frac{1}{2}}$ for $0 \le i \le r$, by

$$\begin{aligned} P_0 &= \boxed{1}\boxed{2}\boxed{\quad\cdots\quad}\boxed{n}, \\ P_{i+\frac{1}{2}} &= P_{i-1} \xrightarrow{\text{jdt}} a_i, \qquad 0 \le i \le r-1, \\ P_{i+1} &= P_{i+\frac{1}{2}} \xleftarrow{\text{RSK}} a_i, \qquad 0 \le i \le r-1, \end{aligned} \tag{2.1}$$

where this notation means that we first remove the letter a_i from P_i using Schützenberger r's *jeu-de-taquin* to get a tableau $P_{i+\frac{1}{2}}$, and then we reinsert a_i into $P_{i+\frac{1}{2}}$ using *Robinson–Schensted–Knuth row insertion* to obtain P_{i+1}. See [Sta2, A1.2,7.11] for the definitions of jeu-de-taquin and RSK insertion. Example 1 provides an example of the application of this algorithm.

For $0 \le i \le k$, let $\lambda^{(i)}$ be the partition shape of the tableau P_i and let $\lambda^{(i+\frac{1}{2})}$ be the partition shape of $P_{i+\frac{1}{2}}$. The final tableau $P_a = P_r$ that results from the insertion of $a = (a_1, \dots, a_r)$ is the *insertion tableau*. It is a standard Young tableaux of shape $\lambda = \lambda^{(r)}$. The sequence of shapes that arise along the way,

$$Q_a = \left((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \dots, \lambda^{(r)} = \lambda \right),$$

is the *recording tableau* of the sequence. The recording tableaux that appear in this process are uniquely described by the following properties:

1. $\lambda^{(0)} = (n)$,

2. For $0 \le i \le r-1$, $\lambda^{(i+\frac{1}{2})}$ is a partition of $n-1$ that is obtained from $\lambda^{(i)}$ by deleting a box,

3. For $1 \le i \le r$, $\lambda^{(i)}$ is a partition of n that is obtained from $\lambda^{(i-\frac{1}{2})}$ by adding a box.

If $\lambda = \lambda^{(r)}$, then a tableau satisfying these properties is called a *r-vacillating tableau of shape* λ. See [HL] and [CDDSY]. The partition shapes that appear in the ℓth step in the process of inserting $a \in \{1, \dots n\}^r$ are in the set

$$\begin{aligned} \Lambda_n^{\ell} &= \left\{ \lambda = (\lambda_1, \dots, \lambda_t) \vdash n \;\middle|\; \lambda_2 + \cdots + \lambda_t \le \ell \right\}. \\ \Lambda_n^{r+\frac{1}{2}} &= \left\{ \lambda = (\lambda_1, \dots, \lambda_t) \vdash n-1 \;\middle|\; \lambda_2 + \cdots + \lambda_t \le k \right\}. \end{aligned} \tag{2.2}$$

The r-vacillating tableau also appear in the *partition algebra lattice*, shown in Figure 2.1 for for $n = 6$ and $0 \le r \le 3$. The partition algebra lattice has vertices on level r given by Λ_n^r. The edges between level r and $r + \frac{1}{2}$ are given by removing a box and the edges between levels $r + \frac{1}{2}$ and $r + 1$ are given by adding a box. Thus, the paths to λ on level r are the r-vacillating tableaux of shape λ. The number of r-vacillating tableaux m_r^λ is the number of paths from the top of the lattice to λ, and we label λ with m_r^λ. We refer to these paths as "tableaux" since they determine paths in the partition algebra lattice in the same way that standard Young tableaux determine paths in Young's lattice.

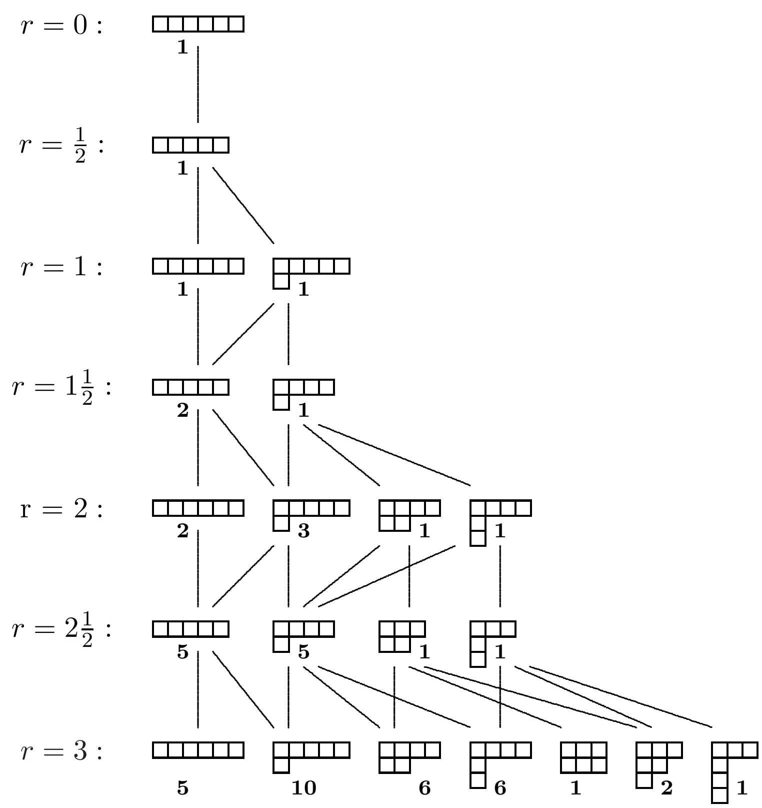


Figure 1: The partition algebra lattice for $n = 6$ and for $0 \le r \le 3$. The rows contain the partitions in Λ_n^r, the paths from the top of the diagram to $\lambda \in \Lambda_n^r$ are the r-vacillating tableaux of shape λ, and the number of paths to λ is m_r^λ. The label on vertex λ is m_r^λ.

We let $a \xrightarrow{\text{DI}} (P_a, Q_a)$ denote the "delete-insert" process defined in (2.1), which associates each $a \in \{1, \dots, n\}^r$ with a pair (P_a, Q_a) consisting of a standard tableau P_a and an r-vacillating tableaux Q_a, each of shape $\lambda \in \Lambda_n^r$. In [HL] this algorithm is shown to be invertible and thus provides a bijection

$$\{1,\dots,n\}^r \xrightarrow{\text{DI}} \bigsqcup_{\lambda \in \Lambda_n^r} \left\{ (P,Q) \;\middle|\; \begin{array}{l} P \text{ is a standard tableau of shape } \lambda \\ Q \text{ is a } r\text{-vacillating tableau of shape } \lambda \end{array} \right\}. \tag{2.3}$$

This gives a combinatorial proof of the identity

$$n^r = \sum_{\lambda \in \Lambda_n^r} f_n^\lambda m_r^\lambda, \tag{2.4}$$

where f^λ is the number of standard tableaux of shape λ (given by the hook formula), and m_r^λ is the number of r-vacillating tableaux of shape λ.

2.2 Delete/Insertion and Major Index

We now show that the bijection (2.1) carries the backsteps associated to integer sequences to the descent set on standard tableaux. We first map sequences in $\{1,\dots,n\}^r$ to permutations in S_n using following surjection

$$\begin{array}{rcl} \{1,\dots,n\}^r & \to & S_n \\ a=(a_1,\dots,a_r) & \mapsto & w_a = \mathrm{RT}(1,2,\dots,n,a_1,\dots,a_r) \end{array} \tag{2.5}$$

where $\mathrm{RT}(1,2,\dots,n,a_1,\dots,a_r)$ is the permutation consisting of the rightmost occurrence of each integer in $\{1,\dots,n\}$. For example,

$$a=(2,1,3,1,6,4,6,3,4) \quad \mapsto \quad w_a = \mathrm{RT}(1,2,3,4,5,6,2,1,3,1,6,4,6,3,4) = (5,2,1,6,3,4).$$

Alternatively, we can produce $w_a = (b_1,b_2,\dots,b_n)$ iteratively using the following algorithm.

$$\begin{array}{ll} (1) & w^{(0)} = (1,2,\dots,n), \\ (2) & \begin{array}{l} w^{(i+1)} \text{ is obtained from } w^{(i)} \text{ by deleting } a_i \text{ from } w^{(i)} \\ \text{and then appending } a_i \text{ to the right of } w^{(i)}, \end{array} \qquad 1 \le i < r. \\ (3) & w_a = w^{(r)}. \end{array} \tag{2.6}$$

Applying this algorithm to $a=(2,1,3,1,6,4,6,3,4)$, for example, yields the same w_a as above:

$$\begin{array}{ll} w^{(0)} = (1,2,3,4,5,6) & w^{(5)} = (4,5,2,3,1,6) \\ w^{(1)} = (1,3,4,5,6,2) & w^{(6)} = (5,2,3,1,6,4) \\ w^{(2)} = (3,4,5,6,2,1) & w^{(7)} = (5,2,1,6,4,3) \\ w^{(3)} = (4,5,6,2,1,3) & w^{(8)} = (5,2,1,6,3,4) \\ w^{(4)} = (4,5,6,2,3,1) & w_a = (5,2,1,6,3,4). \end{array}$$

It is clear that the processes defined in (2.5) and (2.6) yield the same result since the elements of a are cycled to the right end of of w_a in the order that they appear in a..

The *backsteps* (see for example [Lo]) in a permutation $w=(w_1,w_2,\dots,w_n) \in S_n$ are

$$BS(w) = \{\ i \mid i+1 \text{ is to the left of } i \text{ in } w=(w_1,w_2,\dots,w_n)\ \}. \tag{2.7}$$

The *descent set* in $w \in W_n$ is defined by $Des(w) = Des(w_1,w_2,\dots,w_n) = \{\ i \mid w_i > w_{i+1}\ \}$, and it is easy to check that $BS(w) = Des(w^{-1})$. For example if $w=(5,2,1,6,3,4)$ then $BS(w) = Des(w^{-1}) = \{1,4\}$. If P is a standard tableau, then the descent set of P is

$$Des(P) = \{\ i \mid\ i+1 \text{ is in a lower row than } i \text{ in } P\ \}. \tag{2.8}$$

For example $Des\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 9 & 10 \\ \hline 4 & 8 \\ \cline{1-2}\end{array}\right) = \{2,3,6,7\}$. See Example 1 for an illustration of the following proposition.

Proposition 2.1. *If $a \in \{1,\dots,n\}^r$ and $a \xrightarrow{\text{DI}} (P_a, Q_a)$, where P_a is a standard tableau of shape $\lambda \in \Lambda_n^r$ and Q_a is an r-vacillating tableau, then*

$$BS(w_a) = Des(P_a).$$

Proof. The proof is by induction on the length r of $a = (a_1, \dots, a_r)$. If $r = 0$, then $w = \emptyset$ and $P_a = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline\end{array}$. In this case, $w_a = (1, 2, \dots, n)$ has $BS(w_a) = \emptyset = Des(P_a)$.

Now let $r > 0$ and $(a_1, \dots, a_{r-1}) \xrightarrow{\text{DI}} (P_{r-1}, Q_{r-1})$. Then $P_a = (P_{r-1} \xrightarrow{\text{jdt}} a_r) \xleftarrow{\text{RSK}} a_r$, and by induction $Des(P_{r-1}) = BS(w_{(a_1,\dots,a_{r-1})})$. By (2.6), the permutation $w_{(a_1,\dots,a_r)}$ is the same as $w_{(a_1,\dots,a_{r-1})}$ except that it has a_r moved to the the rightmost postition. Since a_r is now to the right of both $a_r - 1$ and $a_r + 1$, and this is the only changed made, we know that

(a) $(a_r - 1, a_r)$ is not a backstep in w_a,

(b) $(a_r, a_r + 1)$ is a backstep in w_a, and

(c) all other $(i, i+1)$ relationships are the same in w_a as they were in $w_{(a_1,\dots,a_{r-1})}$.

These same relationships happen in P:

(a') When a_r is deleted from P_{r-1} (via jeu-de-taquin) and then reinserted (via RSK), it ends up in the first row of P. Thus $(a_r - 1, a_r)$ is not a descent in P.

(b') If $a_r + 1$ was in the first row of P_{r-1} then a_r bumps it to a lower row. Otherwise, it was already in a lower row, and either way $(a_r, a_r + 1)$ is a descent in P.

(c') Whenever i gets bumped into the next row, if $i + 1$ is in that row, i will bump $i + 1$ into a lower row. So if $(i, i+1)$ is a descent it will remain a descent. If $(i, i+1)$ is not a descent, then we must consider the case when $i + 1$ gets bumped lower than i. This only happens if i and $i + 1$ are in the same row. But in this case a number that might bump $i + 1$ would have to be lower than $i + 1$ and thus lower than i. So it might potentially bump i but it would not bump $i + 1$.

It follows by induction that $Des(P_a) = BS(w_a)$, as desired. □

The inverse major index *imaj* of a permutation $w \in S_n$ is the sum of the backsteps in w, and the major index *maj* of a standard tableau P is the sum of the descents in P. That is,

$$imaj(w) = \sum_{i \in BS(w)} i \qquad \text{and} \qquad maj(P) = \sum_{i \in Des(P)} i. \tag{2.9}$$

Note that the major index of w is $maj(w) = \sum_{i \in Des(w)} i$, and $imaj(w) = maj(w^{-1})$. Let q be an indeterminate (in Section 3 we will specialize q to be a prime power). For $\lambda \vdash n$, a q-analog of the hook number f_n^λ is given by

$$f_n^\lambda(q) = \sum_T q^{maj(T)}, \tag{2.10}$$

where the sum is over all standard tableaux T of shape λ. Then $f_n^\lambda(q)$ is the dimension of the irreducible unipotent $\mathrm{GL}_n(\mathbb{F}_q)$-module labeled by λ and it is also given by the q-hook formula (see [Mac, IV.6.7]).

Corollary 2.2. *For all $n, r \in \mathbb{Z}_{>0}$, we have*

$$\sum_{a \in \{1,\ldots,n\}^r} q^{imaj(w_a)} = \sum_{\lambda \in \Lambda_n^r} \sum_{(P,Q)} q^{maj(P)} = \sum_{\lambda \in \Lambda_n^r} f_n^\lambda(q) m_r^\lambda,$$

where (P, Q) ranges over all pairs consisting of a standard tableau P of shape λ and an r-vacillating tableau Q of shape λ, and w_a is defined in (2.5).

Proof. The first equality follows immediately from the fact that the delete-insert bijection (2.3) pairs $a \in \{1, \ldots n\}^r$ with $\{(P_a, Q_a)\}$ and carries $q^{imaj(w_a)}$ to $q^{maj(P)}$. The second equality follows from (2.10) and from the fact that m_r^λ equals the number of r-vacillating tableaux Q of shape λ. ☐

Example 1. The following table illustrates the process of delete-inserting the sequence $a = (3, 5, 2, 3, 2) \in \{1, \ldots, 6\}^5$ to produce a pair (P_a, Q_a) of shape $\lambda = (2, 2, 1, 1)$. The reader should observe that at each step in this process the backsteps in w_a equal the descents in P_a.

i	a_i	P_a		a	w_a	$BS(w_a) = Des(P_a)$
0		1 2 3 4 5 6		$\emptyset$	$(1,2,3,4,5,6)$	$\emptyset$
$\frac{1}{2}$	3	1 2 4 5 6	$\xrightarrow{\text{jdt}}$ 3			
1		1 2 3 5 6 4	$\xleftarrow{\text{RSK}}$ 3	(3)	$(1,2,4,5,6,3)$	$\{3\}$
$1\frac{1}{2}$	5	1 2 3 6 4	$\xrightarrow{\text{jdt}}$ 5			
2		1 2 3 5 4 6	$\xleftarrow{\text{RSK}}$ 5	$(3,5)$	$(1,2,4,6,3,5)$	$\{3,5\}$
$2\frac{1}{2}$	2	1 3 5 4 6	$\xrightarrow{\text{jdt}}$ 2			
3		1 2 5 3 6 4	$\xleftarrow{\text{RSK}}$ 2	$(5,3,2)$	$(1,4,6,3,5,2)$	$\{2,3,5\}$
$3\frac{1}{2}$	3	1 2 5 4 6	$\xrightarrow{\text{jdt}}$ 3			
4		1 2 3 4 5 6	$\xleftarrow{\text{RSK}}$ 3	$(3,5,2,3)$	$(1,4,6,5,2,3)$	$\{3,5\}$
$4\frac{1}{2}$	2	1 3 4 5 6	$\xrightarrow{\text{jdt}}$ 2			
5		1 2 3 5 4 6	$\xleftarrow{\text{RSK}}$ 2	$(3,5,2,3,2)$	$(1,4,6,5,3,2)$	$\{2,3,5\}$

2.3 Set Partitions and Major Index

For an integer $i \geq 0$, define

$$[i] = \frac{q^i - 1}{q - 1} = q^{n-1} + q^{n-2} + \cdots + 1, \tag{2.11}$$

so that $[i]_{q=1} = i$. Recall that the *Stirling number* $S(r, \ell)$ is the number of set partitions of a set of size r into ℓ subsets. We now compute the sum that appears in Corollary 2.2.

Proposition 2.3. *For* $r, n \in \mathbb{Z}_{>0}$,

$$\sum_{a \in \{1,\ldots,n\}^r} q^{imaj(w_a)} = \sum_{\ell=1}^{n} S(r,\ell)[n][n-1]\cdots[n-\ell+1].$$

Proof. We begin by classifying the permutations w_a that appear in the sum. For each sequence $a = (a_1, \ldots, a_r) \in \{1, \ldots, n\}^r$ we define *shape*(a) to be the set partition of $\{1, \ldots, r\}$ given by the rule,

$$i \sim j \text{ in } shape(a) \quad \text{if and only if} \quad a_i = a_j \text{ in } a.$$

We also let

$$D_{n,t} = \{\ w = (w_1, \ldots, w_n) \in S_n \mid w_1 < w_2 < \ldots < w_t\ \},$$

be a distinct set of minimal-length coset representatives of S_n/S_t, where we naturally embed $S_t \subseteq S_n$ as the permutations of $\{1, \ldots, t\}$. From this construction, we immediately have,

$$\text{if} \quad shape(a) \text{ has } \ell \text{ parts} \quad \text{then} \quad w_a \in D_{n,n-\ell}.$$

For example, if $n = 6$, $r = 10$, and $a = (2, 1, 3, 1, 6, 2, 6, 1, 3, 1)$, then

$$a = (\underbrace{2,1,3,1,6,\mathbf{2},\mathbf{6},1,\mathbf{3},\mathbf{1}}_{\ell=4 \text{ distinct entries}}) \qquad \Rightarrow \qquad w_a = (\underbrace{4,5}_{n-\ell=2}, \underbrace{2,6,3,1}_{\ell=4}) \in D_{6,2},$$

$$shape(a) = shape(2,1,3,1,6,2,6,1,3,1) = \underbrace{\{1,6\} \cup \{2,4,8,10\} \cup \{3,9\} \cup \{5\}}_{\ell=4 \text{ parts}}.$$

Note that the number of possible parts ℓ in *shape*(a) is bounded both by the number r of subscripts and the number n of possible choices of a_i.

For a fixed set partition K with ℓ parts and a fixed permutation $w \in D_{n,n-\ell}$ we can easily reconstruct the unique sequence $a \in \{1, \ldots, n\}^r$ with ℓ distinct entries such that *shape*$(a) = K$ and $w_a = w$. Thus, if we let $\mathcal{P}_r^\ell$ be the set partitions of $\{1, \ldots r\}$ with ℓ parts, then

$$\sum_{a \in \{1,\ldots,r\}^r} q^{imaj(w_a)} = \sum_{\ell=1}^{\min(r,n)} \sum_{K \in \mathcal{P}_r^\ell} \sum_{\substack{a \in \{1,\ldots,r\}^r \\ shape(a)=K}} q^{imaj(w_a)} = \sum_{\ell=1}^{\min(n,r)} \sum_{K \in \mathcal{P}_r^\ell} \sum_{w \in D_{n,n-\ell}} q^{imaj(w)}$$

$$= \sum_{\ell=1}^{\min(n,k)} S(r,\ell) \sum_{w \in D_{n,n-\ell}} q^{imaj(w)},$$

where the last equality comes from the fact that the Stirling number $S(r, \ell)$ is the number of partitions of $\{1, \ldots, r\}$ into ℓ parts.

To finish the proof of the proposition, we will show that

$$\sum_{w\in D_{n,t}} q^{imaj(w)} = [n][n-1]\cdots[t+1], \qquad 0 \le t < n. \tag{2.12}$$

The shape of a permutation w is the composition $\mu = (\mu_1, \dots, \mu_\ell)$ of n where μ_1 is the first position i where $w_i > w_{i+1}$, $\mu_1 + \mu_2$ is the next position i where $w_i > w_{i+1}$ and so on. The sum in (2.12) is over all partitions whose shape μ satisfies $\mu_1 \ge t$. An inversion in a permutation w is a pair (i, j) such that $i < j$ and $w_i > w_j$ and $inv(w)$ is the number of inversions in w. Foata and Schützenberger [FS] (see also [Lo, Theorem 11.4.4]) prove that the number of permutations of shape μ having m inversions equals the number of permutations of shape μ having m backsteps. Thus,

$$\sum_{w\in D_{n,t}} q^{imaj(w)} = \sum_{w\in D_{n,t}} q^{inv(w)}. \tag{2.13}$$

Now, our coset representatives $D_{n,t}$ for S_n/S_t are chosen with minimal length, so if $u \in D_{n,t}$ and $v \in S_t$, then $inv(uv) = inv(u) + inv(v)$. Thus,

$$[n]! = \sum_{s\in S_n} q^{inv(w)} = \sum_{u\in D_{n,t}} \sum_{v\in S_t} q^{inv(uv)} = \sum_{u\in D_{n,t}} q^{inv(u)} \sum_{v\in S_t} q^{inv(v)} = \sum_{u\in D_{n,t}} q^{inv(u)}[t]!,$$

where the first and last equalities come from the well-known result of MacMahon (see [Sta1, Cor 1.3.10]) that $\sum_{w\in S_n} q^{inv(w)} = [n]!$. Equation (2.12) follows by dividing by $[t]!$ and replacing inv with $imaj$. □

For $n, r \in \mathbb{Z}_{>0}$, define,

$$d_{n,r}(q) = \sum_{\ell=1}^{n} S(r,\ell)[n][n-1]\cdots[n-\ell+1]. \tag{2.14}$$

The first few values of $d_{n,r}(q)$, for increasing r, are given by

$$\begin{aligned}
d_{n,0}(q) &= 1,\\
d_{n,1}(q) &= [n],\\
d_{n,2}(q) &= [n]([n-1]+1),\\
d_{n,3}(q) &= [n](1+3[n-1]+[n-1][n-2]),\\
d_{n,4}(q) &= [n](1+7[n-1]+6[n-1][n-2]+[n-1][n-2][n-3]).
\end{aligned}$$

When $q = 0$, we have $[j]_{q=0} = 1$, so $d_{n,r}(0) = \sum_{\ell=0}^{n} S(r,\ell)$, which equals the rth Bell number $B(r)$ if $n \ge r$ and which is the number of set partitions of $\{1, \dots, r\}$ into at most n subsets if $n < r$. When $q = 1$ the sum in Proposition 2.3 shows that $d_{n,r}(1)$ equals the cardinality of $\{1, \dots, n\}^k$, so $d_{n,r}(1) = n^k$. These polynomials are tantalizingly close to those in the following identity of Garsia and Remmel [GR, I.17]

$$\sum_{\ell=1}^{r} S(r,\ell;q)[n][n-1]\cdots[n-r+1] = [n]^r,$$

where $S(r,\ell;q)$ is a q-analog of the Stirling number $S(r,\ell)$. Like $d_{n,r}(q)$, these Garsia–Remmel polynomials specialize at $q = 1$ to n^r, but they are different at $q = 0$, since $[n]^k_{q=0}|_{q=0} = 1$.

The next Corollary follows immediately from Corollary 2.2 and Proposition 2.3.

Corollary 2.4. *For $n, r \in \mathbb{Z}_{>0}$, we have*

$$d_{n,r}(q) = \sum_{\ell=1}^{n} S(r,\ell)[n][n-1]\cdots[n-\ell+1] = \sum_{\lambda\in\Lambda_k^n} f_n^\lambda(q) m_k^\lambda.$$

3 A Basis for the $\mathcal{IR}$ Module for $\mathrm{GL}_n(\mathbb{F}_q)$

In this section we construct a module $\mathcal{IR}_q^r$ for the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ using r iterations of Harish-Chandra restriction and induction. We find a basis for $\mathcal{IR}_q^r$ that is indexed by q-set partitions of $\{1,\ldots,r\}$. It is easy to see that the number of these is the polynomial $d_{n,r}(q)$ which appeared in Section 2, and so $\dim(\mathcal{IR}_q^r) = d_{n,r}(q)$. The module $\mathcal{IR}_q^r$ is the defining space for the q-partition algebra, which will be analyzed in a subsequent paper by T. Halverson, A. Ram, and N. Thiem.

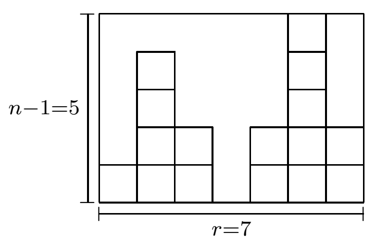
3.1 A family of q-analogues to set partitions

The dimensions of the modules $\mathcal{IR}_q^r$ suggest a q-analogue of set partitions. This section explores some of the combinatorics associated with these objects.

Let

$$\mathbb{Z}_n^r = \{(k_1,k_2,\ldots,k_r)\in\mathbb{Z}^r \mid 0\le k_1,k_2,\ldots,k_r\le n-1\},$$

which we can think of as a configuration of boxes stacked into an $(n-1)\times r$ rectangle. That is, $(k_1,k_2,\ldots,k_r)$ denotes the collection of boxes with k_j boxes stacked in the jth column. For example,



$\leftrightarrow (1,4,2,0,2,5,2).$

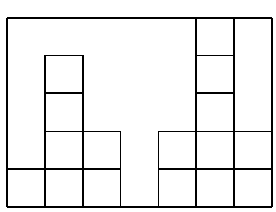
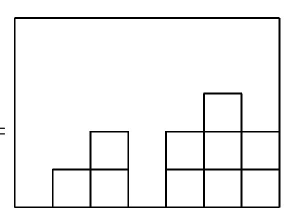
Let

$$\mathcal{P}_{n\times r} = \{(k_1,k_2,\ldots,k_r)\in\mathbb{Z}_n^r \mid\ h_1 = 0, k_j = h \text{ implies } k_i = h-1 \text{ for some } i<j\}.$$

We have a surjection

$$\begin{array}{ccc} \mathbb{Z}_n^r & \longrightarrow & \mathcal{P}_{n\times r} \\ k & \mapsto & k^*, \end{array} \qquad \text{where } k_1^* = 0 \text{ and } k_j^* = \min\left\{k_j, \max\{k_1^*+1, k_2^*+1,\ldots,k_{j-1}^*+1\}\right\},$$

which sends

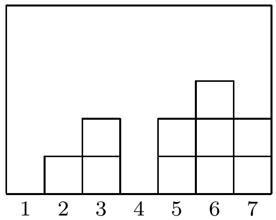
$k =$  $\longmapsto k^* =$ 

We will refer to the heights k^* as the $*$-*height* of k.

There is a bijection,

$$\begin{array}{ccc} \mathcal{P}_{n\times r} & \longleftrightarrow & \left\{\begin{array}{c}\text{Set partitions of } \{1,2,\ldots,r\} \\ \text{with at most } n \text{ parts}\end{array}\right\} \\ k=(k_1,k_2,\ldots,k_r) & \mapsto & K_k, \end{array}$$

where i and j are in the same part of K_k if and only if $k_i = k_j$. That is,

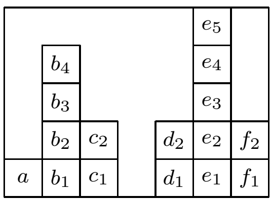


$$\longmapsto \{1,4\} \cup \{2\} \cup \{3,5,7\} \cup \{6\}.$$

To obtain q-analogues, fill the boxes with elements of $\mathbb{F}_q$. Let

$$\mathbb{Z}_n^r(q) = \{((k_1, a^{(1)}), (k_2, a^{(2)}), \ldots, (k_r, a^{(r)})) \mid (k_1, k_2, \ldots, k_r) \in \mathbb{Z}_n^r, a^{(j)} \in \mathbb{F}_q^{k_j}\},$$

For example,

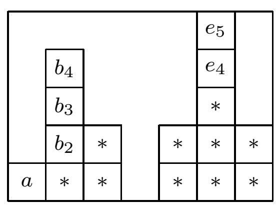


$$\leftrightarrow \left((1, [a]), (4, \begin{bmatrix} b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix}), (2, \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}), (0, ()), (2, \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}), (5, \begin{bmatrix} e_5 \\ e_4 \\ e_3 \\ e_2 \\ e_1 \end{bmatrix}), (2, \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}) \right).$$

The q-analogue of set partitions of $\{1, 2, \ldots, r\}$ with at most n parts is the set

$$\mathcal{P}_{n \times r}(q) = \{((k_1, a^{(1)}), (k_2, a^{(2)}), \ldots, (k_r, a^{(r)})) \mid (k_1, k_2, \ldots, k_r) \in \mathbb{Z}_n^r, a^{(j)} \in \mathbb{F}_q^{k_j - k_j^*}\}.$$

For example,



$$\in \mathcal{P}_{6 \times 7}(q),$$

where the boxes labeled by $*$ give the $*$-height for the associated element in $\mathbb{Z}_n^r$.

An *n-restricted q-set partition of* $\{1, 2, \ldots, r\}$ is an element of $\mathcal{P}_{n \times r}(q)$. Given a set partition K_h with ℓ parts, there are

$$[n][n-1] \cdots [n - \ell + 1]$$

different n-restricted q-set partitions of $\{1, 2, \ldots, r\}$ with $*$-height h. Thus,

$$|\mathcal{P}_{n \times r}(q)| = \sum_{\ell=1}^{n} S(r, \ell)[n][n-1] \cdots [n - \ell + 1] = d_{n,r}(q),$$

where $d_{n,r}(q)$ is defined in (2.14). By the constructions of this section, we also easily obtain the specializations,

$$|\mathcal{P}_{n \times r}(0)| = B(r), \qquad \text{for } n \geq r,$$
$$|\mathcal{P}_{n \times r}(1)| = |\mathbb{Z}_n^r(1)| = n^r,$$

where $B(r)$ is the rth Bell number.

3.2 The Chevalley group $\mathrm{GL}_n(\mathbb{F}_q)$

The general linear group $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ has a double coset decomposition given by

$$G_n = \bigsqcup_{w \in S_n} U_B w B_n, \tag{3.1}$$

where S_n is the subgroup of permutation matrices, and

$$B_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq G_n \quad \text{and} \quad U_B = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subseteq B_n$$

are the subgroups of upper-triangular matrices and unipotent upper-triangular matrices, respectively. For $1 \le i < j \le n$ and $a \in \mathbb{F}_q$, let $x_{ij}(a) \in U_B$ be the matrix with a in the (i,j)th position, ones on the diagonal, and zeroes everywhere else. Note that for $i < j$, $k < l$, $a, b \in \mathbb{F}_q$,

$$x_{ij}(a)x_{kl}(b) = \begin{cases} x_{kl}(b)x_{il}(ab)x_{ij}(a), & \text{if } j = k, \\ x_{kl}(b)x_{kj}(-ab)x_{ij}(a), & \text{if } i = l, \\ x_{ij}(a+b), & \text{if } i = k,\ j = l, \\ x_{kl}(b)x_{ij}(a), & \text{otherwise.} \end{cases} \tag{3.2}$$

For $w \in S_n$, we have

$$x_{ij}(a)w = w x_{w^{-1}(i)w^{-1}(j)}(a). \tag{3.3}$$

Let

$$L_n = \left\{ \left(\begin{array}{c|c} G_1 & 0 \\ \hline 0 & G_{n-1} \end{array} \right) \right\} \subseteq G_n, \quad U_n = \left\{ \left(\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & \mathrm{Id}_{n-1} & \end{array} \right) \right\} \subseteq G_n,$$

and

$$P_n = L_n U_n = \left\{ \left(\begin{array}{c|ccc} G_1 & * & \cdots & * \\ \hline 0 & & G_{n-1} & \end{array} \right) \right\}.$$

For $1 \le k \le n-1$, let

$$w_k = s_k s_{k-1} \cdots s_1,$$

where s_i is the simple reflection that switches i and $i+1$. By convention, $w_0 = 1$. Note that the $w_k, 0 \le k \le n-1$, give a set of minimal-length coset representatives for $S_n/(S_1 \times S_{n-1})$. For $1 \le k \le n-1$ and $a = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_q^k$, let

$$w_k(a) = s_k(a_k)s_{k-1}(a_{k-1}) \cdots s_1(a_1), \qquad \text{where} \qquad s_i(a_i) = x_{i,i+1}(a_i)s_i.$$

Then the decomposition

$$G_n = \bigsqcup_{\substack{0 \le k \le n-1 \\ a \in \mathbb{F}_q^k}} w_k(a) P_n \tag{3.4}$$

follows from (3.1) and (3.3).

3.3 Harish-Chandra Restriction and Induction

To make the notation more manageable, in this section we will assume that n is fixed and drop the subscripts in G_n, P_n, U_n, L_n. Let

$$e_U = \frac{1}{|U|}\sum_{u\in U} u,$$

so that $xe_U = e_U = e_U x$ for all $x \in U$. Since U is a normal subgroup in P, there is a surjection $P \to P/U \cong L$, which gives rise to adjoint functors, called inflation and deflation, respectively,

$$\begin{array}{rccc} \mathrm{Inf}_L^P: & \{\text{Left } L\text{-modules}\} & \longrightarrow & \{\text{Left } P\text{-modules}\}, \\ & V & \mapsto & e_U V \\ \mathrm{Def}_L^P: & \{\text{Left } P\text{-modules}\} & \longrightarrow & \{\text{Left } L\text{-modules}\}. \\ & V & \mapsto & e_U V \end{array}$$

By composing with induction and restriction, we obtain two functors

$$\begin{array}{rccccc} \mathrm{Indf}_L^P: & \{\text{Left } L\text{-modules}\} & \longrightarrow & \{\text{Left } P\text{-modules}\} & \longrightarrow & \{\text{Left } G\text{-modules}\}, \\ & V & \mapsto & e_U V & \mapsto & \mathbb{C}G \otimes_{\mathbb{C}P} e_U V \\ \mathrm{Resf}_L^P: & \{\text{Left } G\text{-modules}\} & \longrightarrow & \{\text{Left } P\text{-modules}\} & \longrightarrow & \{\text{Left } L\text{-modules}\}. \\ & V & \mapsto & V & \mapsto & e_U V \end{array}$$

Let $\mathbb{1}$ denote the trivial module of G. Define the G-module

$$\mathcal{IR}_q^r = \left(\mathrm{Indf}_L^G \mathrm{Resf}_L^G\right)^r(\mathbb{1}), \qquad \text{for } r \geq 0, \tag{3.5}$$

and the L-module

$$\mathcal{IR}_q^{r+\frac{1}{2}} = \mathrm{Resf}_L^G\left(\mathrm{Indf}_L^G \mathrm{Resf}_L^G\right)^r(\mathbb{1}), \qquad \text{for } r \geq 0. \tag{3.6}$$

3.4 A Basis for $\mathcal{IR}_q^r$

Let

$$\otimes_U = \otimes_{\mathbb{C}P} e_U$$

denote tensoring over $\mathbb{C}P$ and multiplying by e_U. By construction it is clear that

$$\begin{aligned} \mathcal{IR}_q^r &= \mathbb{C}\text{-span}\{g_1 \otimes_U g_2 \otimes_U \cdots \otimes_U g_r \otimes \mathbb{1} \mid g_1, g_2, \ldots, g_r \in G\} \\ &= \mathbb{C}\text{-span}\{w_{k_1}(a^{(1)}) \otimes_U \cdots \otimes_U w_{k_r}(a^{(r)}) \otimes \mathbb{1} \mid 0 \leq k_1, \ldots, k_r \leq n-1, a^{(m)} \in \mathbb{F}_q^{k_m}\}, \\ \mathcal{IR}_q^{r+1/2} &= \mathbb{C}\text{-span}\{e_U g_1 \otimes_U g_2 \otimes_U \cdots \otimes_U g_r \otimes \mathbb{1} \mid g_1, g_2, \ldots, g_r \in G\} \\ &= \mathbb{C}\text{-span}\{e_U w_{k_1}(a^{(1)}) \otimes_U \cdots \otimes_U w_{k_r}(a^{(r)}) \otimes \mathbb{1} \mid 0 \leq k_1, \ldots, k_r \leq n-1, a^{(m)} \in \mathbb{F}_q^{k_m}\}. \end{aligned}$$

However, these sets are generally not linearly independent. The following lemma characterizes when two vectors are equal.

Lemma 3.1. *Fix $0 \leq k_1, k_2, \ldots, k_l \leq n-1$ and $a^{(m)} \in \mathbb{F}_q^{k_m}$. Let $1 \leq i < j < n$ and $t \in \mathbb{F}_q^\times$. Then*

$$w_{k_1}(a^{(1)}) \otimes_U \cdots \otimes_U w_{k_l}(a^{(l)}) \otimes_U x_{ij}(t) = w_{k_1}(a^{(1)}) \otimes_U \cdots \otimes_U w_{k_l}(a^{(l)}) \otimes_U 1 \tag{3.7}$$

if and only if $i = 1$ or there exists $1 < m \leq l$ such that

$$w_{k_m} w_{k_{m+1}} \cdots w_{k_l} \quad \textit{sends } i \textit{ to } 1.$$

Proof. Note that for $i=1$, $e_U x_{1j}(t) = e_U$ and for $i>1$, $e_U x_{ij}(t) = x_{ij}(t)e_U$ (since U is normal in U_B). It therefore follows from (3.2) and (3.3) that for $0 \le k \le n-1$ and $a \in \mathbb{F}_q^k$,

$$w_k(a) \otimes_U x_{ij}(t) = \begin{cases} x_{ij}(t) w_k(a) \otimes_U 1, & \text{if } k+1 < i < j, \\ x_{i-1,j}(t) w_k(a) \otimes_U 1, & \text{if } 1 < i \le k+1 < j, \\ x_{i-1,j-1}(t) x_{i-1,k+1}(-a_{j-1}t) w_k(a) \otimes_U 1, & \text{if } 1 < i < j \le k+1, \\ w_k(a) \otimes_U 1, & \text{if } i = 1. \end{cases} \tag{3.8}$$

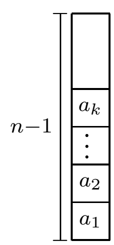
If $i=1$, then (3.7) follows. If $i>1$ and there exists $1<m<l$ such that $w_{k_m} w_{k_{m+1}} \cdots w_{k_l}$ sends i to 1, then

$$e_U w_{k_m}(a^{(m)}) \otimes_U \cdots \otimes_U x_{k_l}(a^{(l)}) \otimes_U x_{ij}(t) = e_U x_{1j'}(t) u w_{k_m}(a^{(m)}) \otimes_U \cdots \otimes_U x_{k_l}(a^{(l)}) \otimes_U 1,$$

for some $j'>1$ and $u \in U$. But $e_U x_{1j'}(t)u = e_U$, giving (3.7).

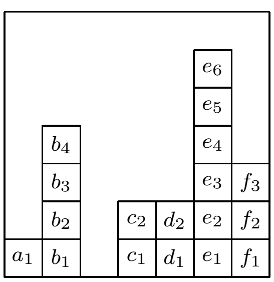
Conversely, (3.8) implies if (3.7) holds then either $i=1$ or there must be an appropriate m. $\square$

Combinatorially, we associate a column of labeled boxes to $w_k(a)$,



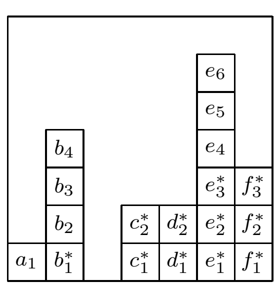
$$\longleftrightarrow \quad w_k(a) = s_k(a_k) \cdots s_2(a_2) s_1(a_1). \tag{3.9}$$

We obtain vectors in $\mathcal{IR}_q^r$ by labeling r stacks of boxes. For example,

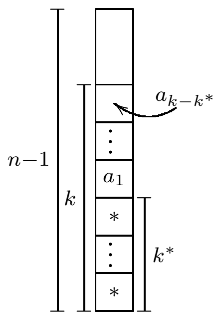


$$\longleftrightarrow \quad w_1(a) \otimes_U w_4(b) \otimes_U w_0 \otimes_U w_2(c) \otimes_U w_2(d) \otimes_U w_6(e) \otimes_U w_3(f) \otimes \mathbb{1}.$$

Lemma 3.1 implies that not all choices of the vectors a, b, c, d, e, f will give different basis vectors of W_q^7. In our example, any change to the $*$-ed values in



will not change the vector in $\mathcal{IR}^7$. That is, given an element of $\mathbb{Z}_n^r(q)$, Lemma 3.1 implies that the $*$-height determines which entries can have arbitrary values (see Section 3.1). For these entries we average over all possible choices. Thus, for each element in $\mathcal{P}_{n\times r}(q)$, we obtain a basis vector. Specifically, for $0 \le k^* \le k \le n-1$ and $a = (a_1, \cdots, a_{k-k^*}) \in \mathbb{F}_q^{k-k^*}$, associate



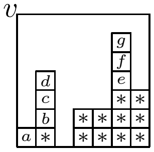
$$\longleftrightarrow \qquad w_k(a) = s_k(a_{k-k^*}) \cdots s_{k^*+1}(a_1) \frac{1}{q^{k^*}} \sum_{b \in \mathbb{F}_q^{k^*}} s_{k^*}(b_{k^*}) \cdots s_1(b_1), \qquad (3.10)$$

For $K = ((k_1, a^{(1)}), \ldots, (k_r, a^{(r)})) \in \mathcal{P}_{n\times r}(q)$, let

$$v_K = w_{k_1}(a^{(1)}) \otimes_U w_{k_2}(a^{(2)}) \otimes_U \cdots \otimes_U w_{k_r}(a^{(r)}) \in \mathbb{C}G \otimes_U \mathbb{C}G \otimes_U \cdots \otimes_U \mathbb{C}G \otimes_U \mathbb{1}_n.$$

For example,

v



$$= x_{12}(a) w_7 \otimes x_{45}(d) x_{35}(c) x_{25}(b) \frac{1}{q} \sum_{t \in \mathbb{F}_q} x_{15}(t) w_4 \otimes 1 \otimes \frac{1}{q^2} \sum_{s,t \in \mathbb{F}_q} x_{23}(s) x_{13}(t) w_6$$

$$\otimes \frac{1}{q^2} \sum_{s,t \in \mathbb{F}_q^\times} x_{23}(s) x_{13}(t) w_6 \otimes x_{67}(g) x_{57}(f) x_{47}(e) \frac{1}{q^3} \sum_{r,s,t \in \mathbb{F}_q} x_{37}(r) x_{27}(s) x_{17}(t) w_2$$

$$\otimes \frac{1}{q^3} \sum_{r,s,t \in \mathbb{F}_q} x_{34}(r) x_{24}(s) x_{14}(t) \otimes \mathbb{1}_n.$$

Lemma 3.1 and the following discussion imply that the v_K are linearly independent, so we have proved the first part of the following theorem.

Theorem 3.2. *Let $r \in \mathbb{Z}_{\ge 0}$. Then*

(a) The G-module $\mathcal{IR}_q^r$ has a basis given by

$$\{v_K \mid K \in \mathcal{P}_{n\times r}(q)\},$$

and thus $\dim(\mathcal{IR}_q^r) = d_{n,r}(q)$.

(b) The L-module $W_q^{r+1/2}$ has a basis given by

$$\{v_K \mid K \in \mathcal{P}_{n\times r+1}(q) \text{ with } k_1 = 0\},$$

and thus $\dim(\mathcal{IR}_q^r) = d_{n,r+1}(q)/[n]$.

To prove Theorem 3.2 (b), it suffices to characterize what happens in $e_U \mathcal{IR}_q^r$. Let

$$\begin{array}{ccc} \mathcal{P}_{n\times r}(q) & \longrightarrow & \{K \in \mathcal{P}_{n\times r+1}(q) \mid k_1 = 0\} \\ K & \mapsto & \tilde{K} \end{array}$$

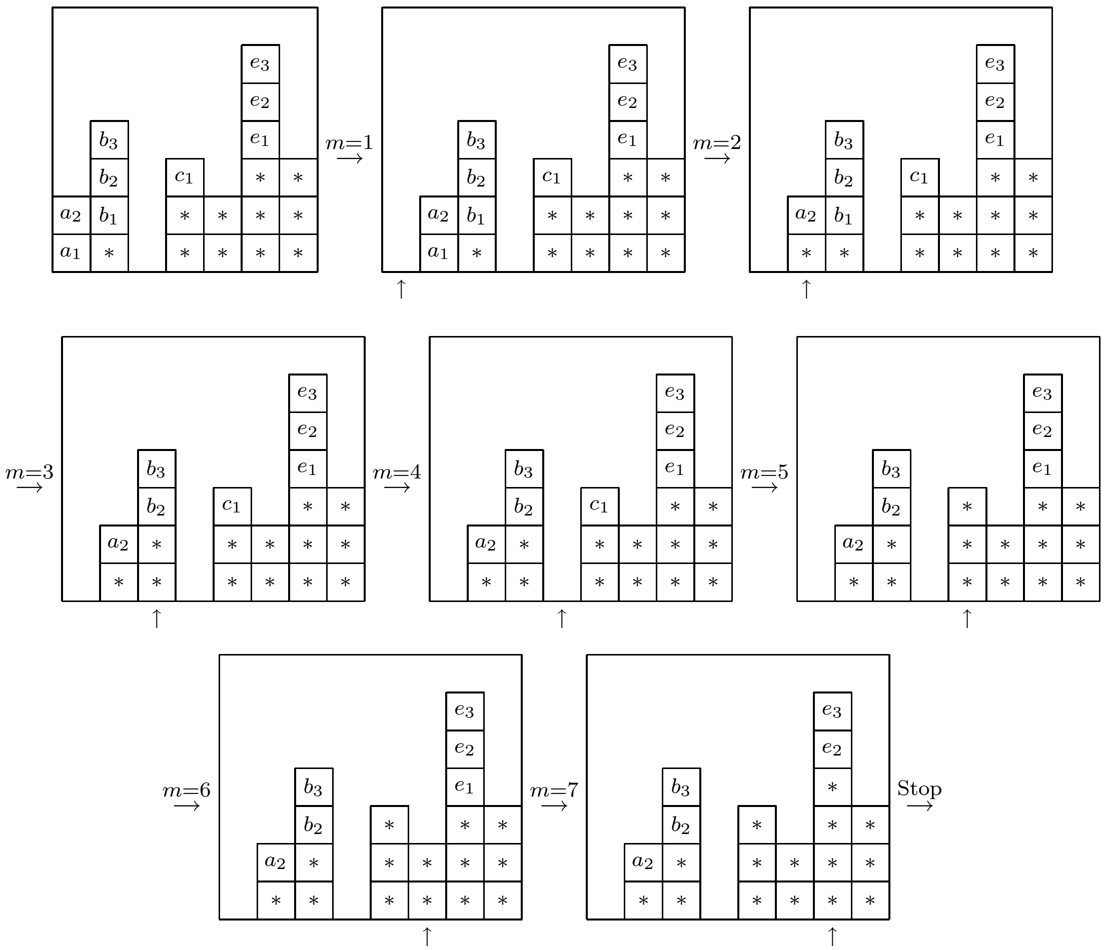
be the surjective function given by the following algorithm.

(1) Add an empty column to the left side of K and set $m = 1$,

(2) If the resulting diagram is in $\mathcal{P}_{n\times r+1}(q)$, stop. Else set $m := m + 1$.

(3) If column m has an unstarred box, then replace the bottom unstarred entry by $*$. Go to step (2).

For example, we get



Lemma 3.3. *Let $K \in \mathcal{P}_{n\times k}(q)$. Then $e_U v_K = v_{\tilde{K}}$.*

Proof. Apply Lemma 3.1 to the vector

$$w_0 \otimes_U v_K$$

to obtain the statement of this lemma. □

Proof of Theorem 3.2 (b). We have that

$$\begin{aligned}\mathcal{IR}_q^{r+1/2} &= e_U \mathcal{IR}_q^r \\ &= \mathbb{C}\text{-span}\{e_U v_K \mid K \in \mathcal{P}_{n\times r}(q)\} \\ &= \mathbb{C}\text{-span}\{v_{\tilde{K}} \mid \mathcal{P}_{n\times r}(q)\},\end{aligned}$$

and by Lemma 3.1, the vectors in the last set are linearly independent. □

4 Group action on $\mathcal{IR}^r$

In general,

$$gw_k(a) \otimes_U v = w_l(b) \otimes_U pv, \qquad \text{where } gw_k(a) = w_l(b)p.$$

Thus, globally the matrix of g is the matrix of g acting by left multiplication on G/P. The group G has generators given by

$$\{x_{ij}(t) \mid 1 \le i < j \le n, t \in \mathbb{F}_q\} \cup \{s_1, s_2, \ldots, s_{n-1}\} \cup \{h_k(t) \mid 1 \le k \le n, t \in \mathbb{F}_q^\times\},$$

where $h_k(t)$ is the identity matrix with the kth diagonal 1 replaced by t. The generators of G act on $\mathcal{IR}^r(\mathbb{1})$ in the following way:

$$s_i w_k(a) \otimes_U v = \begin{cases} w_k(a) \otimes_U s_i v, & \text{if } i > k+1, \\ w_{k+1}(a_1,a_2,\ldots,a_k,0) \otimes_U v, & \text{if } i = k+1, \\ w_{k-1}(a_1,\ldots,a_{k-1}) \otimes_U v, & \text{if } i = k,\ a_k = 0, \\ w_k(a_1,\ldots,a_{k-1},a_k^{-1}) \otimes_U h_{k+1}(-a_k^{-1})v, & \text{if } i = k,\ a_k \ne 0, \\ w_k(a_1,\ldots,a_{i-1},a_{i+1},a_i,a_{i+2},\ldots,a_k) \otimes_U s_{i+1}v, & \text{if } i < k. \end{cases}$$

$$h_j(b) w_k(a) \otimes_U v = \begin{cases} w_k(a) \otimes_U h_j(b)v, & \text{if } j > k+1, \\ w_k(a_1 b^{-1},\ldots,a_k b^{-1}) \otimes_U v, & \text{if } j = k+1, \\ w_k(a_1,\ldots,a_{j-1},ba_j,a_{j+1},\ldots,a_k) \otimes_U h_{j+1}(b)v, & \text{if } i < k+1. \end{cases}$$

$$x_{ij}(b) w_k(a) \otimes_U v = \begin{cases} w_k(a) \otimes_U x_{ij}(b)v, & \text{if } i > k+1, \\ w_k(a) \otimes_U x_{k+1,j}(-a_k b) \cdots x_{2,j}(-a_1 b)v, & \text{if } i = k+1 \ne 1, \\ w_k(a) \otimes_U v, & \text{if } i = k+1 = 1, \\ w_k(a) \otimes_U x_{i+1,j}(b)v, & \text{if } i < k+1 < j, \\ w_k(a_1,\ldots,a_{i-1},a_i+b,a_{i+1},\ldots,a_k) \otimes_U v, & \text{if } i < k+1 = j \\ w_k(a_1,\ldots,a_{i-1},a_i+ba_j,a_{i+1},\ldots,a_k) \otimes_U x_{i+1,j+1}(b)v, & \text{if } j < k+1. \end{cases}$$

References

[CDDSY] W. Chen, E.. Deng, R. Du, R. Stanley, and C. Yan, Crossings and nestings of matchings and partitions, *Trans. Amer. Math. Soc.* **359** (2007), 1555–1575.

[FS] D. Foata and M-P. Schützenberger, Major index and inversion number of permutations, *Math. Nachr.*, **83** (1978), 143–159.

[GR] A. Garsia and J. Remmel, Q-counting rook configurations and a formula of Frobenius, *J. Combin. Theory Ser. A*, **41** (1986), 246–275.

[HL] T. Halverson and T. Lewandowski, RSK insertion for set partitions and diagram algebras *Electron. J. Combin.*, **11** (2004/06), 24.

[HR1] T. Halverson and A. Ram, A q-Partition Algebra, Abstract of Talk at AMS Fall 2004 Central Section Meeting, Abstract Issue 25/4, October 2004.

[HR2] T. Halverson and A. Ram, Partition Algebras, *European J. Combinatorics*, **26** (2005), 869-921.

[Jo] V. F. R. Jones, The Potts model and the symmetric group, in: *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras (Kyuzeso, 1993),* World Sci. Publishing, River Edge, NJ, 1994, 259–267.

[Lo] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, **90** Cambridge University Press, Cambridge, 2002.

[Mac] I. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Univ. Press, New York, 1995.

[Mar1] P. Martin, Temperley-Lieb algebras for nonplanar statistical mechanics—the partition algebra construction, *J. Knot Theory Ramifications* **3** (1994), 51–82.

[Mar2] P. Martin, The structure of the partition algebras, *J. Algebra* **183** (1996), 319–358.

[Sta1] R. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, Cambridge, 1997.

[Sta2] R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.