# A skein-like multiplication algorithm for unipotent Hecke algebras

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**Abstract**

Let $G$ be a finite group of Lie type (e.g. $GL_n(\mathbb{F}_q)$) and $U$ a maximal unipotent subgroup of $G$. If $\psi$ is a linear character of $U$, then the unipotent Hecke algebra is $\mathcal{H}_\psi = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi))$. Unipotent Hecke algebras have a natural basis coming from double cosets of $U$ in $G$. This paper describes relations for reducing products of basis elements, and gives a detailed description of the implications in the case $G = GL_n(\mathbb{F}_q)$.

## 1 Introduction

Unipotent Hecke algebras interpolate between two classical Hecke algebras, the Gelfand-Graev Hecke algebra [St, Yo1] and the Yokonuma algebra [Yo2] (a generalization of the Iwahori-Hecke algebra). These two classical algebras have not generally been studied from the same perspective, and an underlying philosophy of this paper is that techniques employed in the study of one classical algebra not only apply to the other, but also to all unipotent Hecke algebras.

The Gelfand-Graev Hecke algebra is a commutative algebra that has connections with Chevalley group representation theory [DM], unipotent orbits [Ka1], and Kloosterman sums [CS]. Despite being commutative, computing products in the standard double-coset basis is a challenging problem. The definition of a Hecke algebra implies [CR] that if $T_h$ and $T_k$ are two basis elements, then

$$T_k T_h = \sum_v c_{kh}^v T_v, \quad \text{where} \quad c_{kh}^v = \frac{1}{|U|^2} \sum_{\substack{u_1,u_2,u_3,u_4 \in U \\ u_1 k u_2 = v u_3 h^{-1} u_4}} \psi_\mu(u_1^{-1} u_2^{-1} u_3 u_4), \tag{$*$}$$

but this formula is unhelpful for many applications. Using a geometric approach in [Cu], Curtis analyzed which elements appear in the sums of $(*)$, but computing products in the Gelfand-Graev algebra still remains difficult.

This paper provides a uniform solution to the multiplication problem for Yokonuma Hecke algebras, Gelfand-Graev Hecke algebras, and all unipotent Hecke algebras. The idea is that in a unipotent Hecke algebra the $c_{hk}^v$ in $(*)$ are determined by generalizations of the braid-like relations of the Iwahori-Hecke algebra, and that the multiplication in any unipotent Hecke algebra can be done in a manner directly analogous to the way it is done in the Iwahori-Hecke algebra.

Let $G$ be a finite Chevalley group with a maximal unipotent subgroup $U$. Suppose $\psi_\mu : U \to \mathbb{C}^*$ is a linear character of $U$. Then the *unipotent Hecke algebra* $\mathcal{H}(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi_\mu)) \cong e_\mu \mathbb{C}G e_\mu, \qquad \text{where} \quad e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1}) u.$$

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Fix a subgroup $N \subseteq G$ of double coset representatives

$$G = \bigsqcup_{v \in N} UvU, \qquad \text{and let} \qquad N_\mu = \{v \in N \mid e_\mu v e_\mu \neq 0\}.$$

Then the set $\{e_\mu v e_\mu \mid v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$ [CR, Prop. 11.30].

**Examples.**

1. **The Yokonuma Hecke algebra.** If $\psi_\mu = \mathbb{1}$ is the trivial character, then $N_{\mathbb{1}} = N$. Let $W = \langle s_1, s_2, \ldots, s_\ell \rangle$ be the Weyl group of $G$ and $T = \langle h_i(t) \mid 1 \le i \le \ell, t \in \mathbb{F}_q^* \rangle$ be a maximal torus so that $N \cong T \rtimes W$. For $w \in W$ and $h \in T$, let $T_{hw} = e_{\mathbb{1}} hw e_{\mathbb{1}}$. By [Yo2], the Yokonuma algebra $\mathcal{H}_{\mathbb{1}}$ has a basis $\{T_v \mid v \in N\}$ with relations

$$\begin{aligned}
T_{s_i} T_w &= \begin{cases} T_{s_i w}, & \text{if } \ell(s_i w) = \ell(w) + 1, \\ q^{-1} T_{h_i(-1) s_i w} + q^{-1} \sum_{t \in \mathbb{F}_q^*} T_{h_i(t) w}, & \text{if } \ell(s_i w) = \ell(w) - 1, \end{cases} && 1 \le i \le \ell,\ w \in W, \\
T_h T_w &= T_{hw}, && h \in T,\ w \in W, \\
T_h T_k &= T_{hk}, && h, k \in T,
\end{aligned}$$

where if $w = s_{i_1} s_{i_2} \ldots s_{i_r} \in W$ for $r$ minimal, then $\ell(w) = r$. These relations give an "efficient" way to compute arbitrary products $(e_{\mathbb{1}} u e_{\mathbb{1}})(e_{\mathbb{1}} v e_{\mathbb{1}})$ in $\mathcal{H}_{\mathbb{1}}$.

2. **The Gelfand-Graev Hecke algebra.** If $\psi_\mu$ is in general position, then the Gelfand-Graev module $\mathrm{Ind}_U^G(\psi_\mu)$ is multiplicity free as a $G$-module ([Yo1],[St, Theorem 49]). The corresponding Hecke algebra $\mathcal{H}_\mu$ is therefore commutative. However, decomposing the product $(e_\mu u e_\mu)(e_\mu v e_\mu)$ into basis elements is more challenging than in the Yokonuma case [Ch, Cu, Ra].

Section 3 describes some of the subalgebra structure of unipotent Hecke algebras. The main results of the paper are in Section 4:

> Theorem 4.1 and Corollary 3 give relations similar to those of the Yokonuma algebra (example 1, above) for evaluating the product $(e_\mu u e_\mu)(e_\mu v e_\mu)$, with $u, v \in N_\mu$, in any unipotent Hecke algebra $\mathcal{H}_\mu$.

Section 5 applies the main results to the special case when $G = GL_n(\mathbb{F}_q)$, the general linear group over a finite field $\mathbb{F}_q$ with $q$ elements. Readers unfamiliar with the discourse of Chevalley groups may skip ahead to Section 5 (which is independent of Sections 3 and 4).

There are several natural ways to generalize unipotent Hecke algebras. In a series of papers [Ka1, Ka2, Ka3] Kawanaka has analyzed a family of modules obtained by relaxing the maximality condition on $U$. There has also recently been a growing interest in a larger family of characters known as super characters [An, ACDS]. Seeing which aspects of the techniques associated with unipotent Hecke algebras extend to the Hecke algebras of these characters would be an interesting continuation of this work.

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## 2 Preliminaries

### 2.1 Finite Chevalley groups

Let $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_s$ be a reductive Lie algebra, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. If $\mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{g}_s$, then $\mathfrak{h} = Z(\mathfrak{g}) \oplus \mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}) \qquad \text{and} \qquad \mathfrak{h}_s^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_s, \mathbb{C}).$$

As an $\mathfrak{h}_s$-module, $\mathfrak{g}_s$ decomposes

$$\mathfrak{g}_s \cong \mathfrak{h}_s \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}_s)_\alpha, \quad \text{where} \quad (\mathfrak{g}_s)_\alpha = \langle X \in \mathfrak{g}_s \mid [H, X] = \alpha(H)X, H \in \mathfrak{h}_s \rangle,$$

and $R = \{\alpha \in \mathfrak{h}_s^* \mid \alpha \neq 0, (\mathfrak{g}_s)_\alpha \neq 0\}$ is the set of roots of $\mathfrak{g}_s$. Choose a set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. This choice splits the set of roots $R$ into positive roots $R^+$ and negative roots $R^-$ with $R^- = -R^+$.

For each pair of roots $\alpha, -\alpha$, there exists a Lie algebra isomorphism $\phi_\alpha : \mathfrak{sl}_2 \to \langle \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \rangle$. Choose these isomorphisms such that if

$$X_\alpha = \phi_\alpha \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \in (\mathfrak{g}_s)_\alpha, \qquad H_\alpha = \phi_\alpha \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \mathfrak{h}_s, \qquad X_{-\alpha} = \phi_\alpha \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) \in (\mathfrak{g}_s)_{-\alpha},$$

then $\{X_\alpha, H_{\alpha_i} \mid \alpha \in R, 1 \le i \le \ell\}$ is a Chevalley basis of $\mathfrak{g}_s$ [Hu, Theorem 25.2].

Let $V$ be a finite dimensional $\mathfrak{g}$-module such that $V$ has a $\mathbb{C}$-basis $\{v_1, v_2, \ldots, v_r\}$ that satisfies

(a) There exists a $\mathbb{C}$-basis $\{H_1, \ldots, H_n\}$ of $\mathfrak{h}$ such that

 (1) $H_{\alpha_i} \in \mathbb{Z}_{\ge 0}$-span$\{H_1, \ldots, H_n\}$,
 (2) $H_i v_j \in \mathbb{Z} v_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$.
 (3) $\dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}) \le \dim_{\mathbb{C}}(\mathfrak{h})$.

(b) $\dfrac{X_\alpha^n}{n!} v_i \in \mathbb{Z}$-span$\{v_1, v_2, \ldots, v_r\}$ for $\alpha \in R$, $n \in \mathbb{Z}_{\ge 0}$ and $i = 1, 2, \ldots, r$.

(c) $\dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{v_1, v_2, \ldots, v_r\}) \le \dim_{\mathbb{C}}(V)$.

(Condition (a) guarantees that $Z(\mathfrak{g})$ acts diagonally. If $Z(\mathfrak{g}) = 0$, then the existence of such a basis is guaranteed by a theorem of Kostant [Hu, Theorem 27.1]).

Let

$$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}. \tag{2.1}$$

The finite field $\mathbb{F}_q$ with $q$ elements has a multiplicative group $\mathbb{F}_q^*$ and an additive group $\mathbb{F}_q^+$. Let

$$V_q = \mathbb{F}_q\text{-span}\{v_1, v_2, \ldots, v_r\}. \tag{2.2}$$

The *finite reductive Chevalley group*

$$G_V = \langle x_\alpha(a), h_H(b) \mid \alpha \in R, H \in \mathfrak{h}_{\mathbb{Z}}, a \in \mathbb{F}_q, b \in \mathbb{F}_q^* \rangle,$$

is the subgroup of $GL(V_q)$ generated by the elements

$$x_\alpha(a) = \sum_{n \ge 0} a^n \frac{X_\alpha^n}{n!}, \quad \text{and} \tag{2.3}$$

$$h_H(b) = \mathrm{diag}(b^{\lambda_1(H)}, b^{\lambda_2(H)}, \ldots, b^{\lambda_r(H)}), \quad \text{where } Hv_i = \lambda_i(H) v_i. \tag{2.4}$$

**Remark.** If $\mathfrak{g} = \mathfrak{g}_s$, then $G_V = \langle x_\alpha(t) \mid \alpha \in R, t \in \mathbb{F}_q\rangle$.

**Example**. Suppose $\mathfrak{g} = \mathfrak{gl}_2$ and let

$$V = \mathbb{C}\text{-span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

be the natural $\mathfrak{g}$-module $\mathbb{C}^2$ given by matrix multiplication. Then $\mathfrak{h}$ has a basis

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \right\} = \mathbb{C}\text{-span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

By direct computation,

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)}(t) = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \quad \text{for } a, b \in \mathbb{Z},$$

and $G_V = GL_2(\mathbb{F}_q)$ (the general linear group).

## 2.2 Important subgroups of a Chevalley group

Let $G = G_V$ be a Chevalley group defined with a $\mathfrak{g}$-module $V$ as above. The group $G$ contains a subgroup $U$ given by

$$U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q\rangle,$$

which decomposes as

$$U = \prod_{\alpha \in R^+} U_\alpha, \qquad \text{where} \quad U_\alpha = \langle x_\alpha(t) \mid t \in \mathbb{F}_q\rangle,$$

with uniqueness of expression for any fixed ordering of the positive roots [St, Lemma 18]. For each $\alpha \in R^+$, the map

$$\begin{array}{rcl} U_\alpha & \xrightarrow{\sim} & \mathbb{F}_q^+ \\ x_\alpha(t) & \mapsto & t \end{array}$$

is a group isomorphism.

For $\alpha, \beta \in R$, define the maps

$$\begin{array}{rccc} s_\alpha : & \mathfrak{h}^* & \longrightarrow & \mathfrak{h}^* \\ & \gamma & \mapsto & \gamma - \gamma(H_\alpha)\alpha \end{array} \quad \text{and} \quad \begin{array}{rccc} s_\alpha : & \mathfrak{h} = Z(\mathfrak{g}) \oplus \mathfrak{h}_s & \longrightarrow & \mathfrak{h} \\ & H + H_\beta & \mapsto & H + H_\beta - \beta(H_\alpha)H_\alpha \end{array}. \tag{2.5}$$

The *Weyl group of* $G$ is $W = \langle s_\alpha \mid \alpha \in R\rangle$ and has a presentation

$$W = \langle s_1, s_2, \ldots, s_\ell \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \le i \ne j \le \ell\rangle, \qquad m_{ij} \in \mathbb{Z}_{>0},\ s_i = s_{\alpha_i}.$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ with $r$ minimal, then the *length* of $w$ is $\ell(w) = r$.

Let $\mathfrak{h}_\mathbb{Z}$ be as in (2.1). If $q > 3$, then the subgroup

$$T = \langle h_H(t) \mid H \in \mathfrak{h}_\mathbb{Z}, t \in \mathbb{F}_q^*\rangle$$

has its normalizer in $G$ given by

$$N = \langle w_\alpha(t), h \mid \alpha \in R, h \in T, t \in \mathbb{F}_q^*\rangle, \quad \text{where } w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t).$$

If $\alpha \in R$, then $h_{H_\alpha}(t) = w_\alpha(t)w_\alpha(1)^{-1}$. Write $h_\alpha(t) = h_{H_\alpha}(t)$ and $h_i(t) = h_{\alpha_i}(t)$.

There is a natural surjection from $N$ onto the Weyl group $W$ with kernel $T$ given by

$$\begin{array}{rccll} \pi: & N & \longrightarrow & W & \\ & w_\alpha(t) & \mapsto & s_\alpha, & \text{for } \alpha \in R, t \in \mathbb{F}_q^*, \\ & h & \mapsto & 1, & \text{for } h \in T. \end{array} \tag{2.6}$$

Suppose $v \in N$. Then for each minimal expression

$$\pi(v) = s_{i_1} s_{i_2} \ldots s_{i_r}, \quad \text{with } \ell(\pi(v)) = r,$$

there is a unique $v_T \in T$ such that

$$v = w_{i_1}(1) w_{i_2}(1) \cdots w_{i_r}(1) v_T. \tag{2.7}$$

To simplify some notation in later sections, write

$$v = v_1 v_2 \cdots v_r v_T, \qquad \text{where } v_{i_k} = w_{i_k}(1). \tag{2.8}$$

## 2.3 Unipotent Hecke algebras

Let $G$ be a finite Chevalley group. Fix a nontrivial homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$. If

$$\begin{array}{rccc} \mu: & R^+ & \to & \mathbb{F}_q \\ & \alpha & \mapsto & \mu_\alpha \end{array} \qquad \text{satisfies } \mu_\alpha = 0 \text{ for all } \alpha \text{ not simple,} \tag{2.9}$$

then the map

$$\begin{array}{rccc} \psi_\mu: & U & \longrightarrow & \mathbb{C}^* \\ & x_\alpha(t) & \mapsto & \psi(\mu_\alpha t) \end{array} \tag{2.10}$$

is a linear character of $U$. With the exception of a few degenerate special cases of $G$ (which can be avoided if $q > 3$), all linear characters of $U$ are of this form [Yo1.5, Theorem 1].

The *unipotent Hecke algebra* $\mathcal{H}(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi_\mu)), \tag{2.11}$$

or viewed as a subset of $\mathbb{C}G$,

$$\mathcal{H}_\mu = e_\mu \mathbb{C}G e_\mu, \qquad \text{where} \quad e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1}) u. \tag{2.12}$$

**Remark:** Since $T$ is in the normalizer of $U$ in $G$, $T$ acts on the linear characters of $U$ by

$${}^h\chi(u) = \chi(huh^{-1}), \qquad \text{where } u \in U,\ h \in T, \text{ and } \chi : U \to \mathbb{C}^*.$$

If two linear characters $\chi$ and $\gamma$ are in the same $T$-orbit then $\mathcal{H}(G, U, \chi) \cong \mathcal{H}(G, U, \gamma)$ (although the converse does not necessarily hold).

The group $G$ has a double-coset decomposition

$$G = \bigsqcup_{v \in N} UvU, \qquad [\mathrm{St, Theorem\ 4}] \tag{2.13}$$

and if

$$\begin{aligned} N_\mu &= \{v \in N \mid e_\mu v e_\mu \neq 0\} \\ &= \{v \in N \mid u, vuv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vuv^{-1})\} \end{aligned} \tag{2.14}$$

then the set $\{e_\mu v e_\mu \mid v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$ [CR, Prop. 11.30].

**Examples** (see also the Introduction).

1. **The Yokonuma Hecke algebra.** If $\mu_\alpha = 0$ for all positive roots $\alpha$, then $\psi_\mu = \mathbb{1}$ is the trivial character and $N_{\mathbb{1}} = N$. Let $T_v = e_{\mathbb{1}} v e_{\mathbb{1}}$ for $v \in N$, with $T_i = T_{w_i(1)}$ and $T_H(t) = T_{h_H(t)}$. If $v = v_1 v_2 \cdots v_r v_T \in N$ according to a minimal expression $s_{i_1} s_{i_2} \cdots s_{i_r} \in W$ (as in (2.8)), then
$$T_v = T_{i_1} T_{i_2} \cdots T_{i_r} T_{v_T}.$$
Thus, the Yokonuma algebra $\mathcal{H}_{\mathbb{1}}$ has generators $\{T_i,\, T_h \ \mid\ 1 \le i \le \ell,\, h \in T\}$ (see [Yo2]) with relations,
$$\begin{aligned} T_i^2 &= q^{-1} T_{H_{\alpha_i}}(-1) + q^{-1} \sum_{t \in \mathbb{F}_q^*} T_{H_{\alpha_i}}(t^{-1}) T_i, && 1 \le i \le \ell, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij}\text{ terms}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij}\text{ terms}}, && (s_i s_j)^{m_{ij}} = 1, \\ T_i T_h &= T_{s_i h} T_i, && h \in T, \\ T_h T_k &= T_{hk}, && h, k \in T. \end{aligned}$$
These relations give an "efficient" way to compute arbitrary products $T_u T_v$ in $\mathcal{H}_{\mathbb{1}}$. There is a surjective map from the Yokonuma algebra onto the Iwahori-Hecke algebra that sends $T_h \mapsto 1$ for all $h \in T$. "Setting $T_h = 1$" in the Yokonuma algebra relations recovers relations for the Iwahori-Hecke algebra,
$$T_i^2 = q^{-1} + q^{-1}(q-1) T_i, \qquad \underbrace{T_i T_j \cdots}_{m_{ij}\text{ terms}} = \underbrace{T_j T_i \cdots}_{m_{ij}\text{ terms}}.$$
Furthermore, there is a surjective map from the Iwahori Hecke algebra onto the group algebra of the Weyl group given by $T_i \mapsto s_i$ and $q \mapsto 1$. Thus, by "setting $T_i = s_i$ and $q = 1$" we retrieve the Coxeter relations of $W$,
$$s_i^2 = 1, \qquad \underbrace{s_i s_j s_i \cdots}_{m_{ij}\text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\text{ terms}}.$$

2. **The Gelfand-Graev Hecke algebra.** By definition, if $\mu_\alpha \neq 0$ for all simple roots $\alpha$, then $\psi_\mu$ is in general position. The Gelfand-Graev Hecke algebra $\mathcal{H}_\mu$ is commutative ([Yo1],[St, Theorem 49]).

# 3 Parabolic subalgebras of $\mathcal{H}_\mu$

Let $\psi_\mu : U \to G$ be as in (2.10). Fix a subset $J \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ such that
$$J \supseteq \{\alpha_i \text{ simple root } \ \mid\ \mu_{\alpha_i} \neq 0\}. \tag{3.1}$$
For example, if $\psi_\mu$ is in general position, then $J = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, but if $\psi_\mu$ is trivial, then $J$ could be any subset.

Let
$$W_J = \langle s_i \in W \ \mid\ \alpha_i \in J \rangle, \quad P_J = \langle U, T, W_J \rangle \quad \text{and} \quad R_J = \mathbb{Z}\text{-span}\{\alpha_i \in J\} \cap R.$$
Then $P_J$ has subgroups
$$L_J = \langle T, W_J, U_\alpha \ \mid\ \alpha \in R_J \rangle \quad \text{and} \quad U_J = \langle U_\alpha \ \mid\ \alpha \in R^+ - R_J \rangle \tag{3.2}$$

(a Levi subgroup and the unipotent radical of $P_J$, respectively). Note that

$$U_J L_J = P_J, \quad U_J \cap L_J = 1, \quad \text{and, in fact,} \quad P_J = U_J \rtimes L_J.$$

Define the idempotents of $\mathbb{C}U$,

$$e_{\mu J} = \frac{1}{|L_J \cap U|} \sum_{u \in L_J \cap U} \psi_\mu(u^{-1})u \quad \text{and} \quad e'_J = \frac{1}{|U_J|} \sum_{u \in U_J} u, \tag{3.3}$$

so that $e_\mu = e_{\mu J} e'_J$ is the decomposition of $e_\mu$ with respect to $P = L_J U_J$.

The group homomorphisms

$$\begin{array}{ccc} P_J & \longrightarrow & L_J \\ lu & \mapsto & l \end{array} \quad \text{and} \quad \begin{array}{ccc} P_J & \longrightarrow & G \\ lu & \mapsto & lu \end{array} \quad \text{for } l \in L_J,\, u \in U_J,$$

induce functors

$$\begin{array}{rcl} \mathrm{Inf}_{L_J}^{P_J} :\{L_J\text{-modules}\} & \longrightarrow & \{P_J\text{-modules}\} \\ M & \mapsto & e'_J M \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Ind}_{P_J}^{G} :\{P_J\text{-modules}\} & \longrightarrow & \{G\text{-modules}\} \\ M' & \mapsto & \mathbb{C}G \otimes_{\mathbb{C}P_J} M' \end{array}$$

whose composition is the functor $\mathrm{Indf}_{L_J}^G$. In the special case when $(\mathbb{C}L_J)e$ is an $L_J$-module with corresponding idempotent $e$,

$$\begin{array}{rccc} \mathrm{Indf}_{L_J}^G : & \{L_J\text{-modules}\} & \longrightarrow & \{G\text{-modules}\} \\ & \mathbb{C}L_J e & \mapsto & \mathbb{C}Gee'_J. \end{array}$$

The map $\psi_\mu : U \to \mathbb{C}^*$ restricts to a linear character $\mathrm{Res}^U_{U\cap L_J}(\psi_\mu) : L_J \cap U \to \mathbb{C}^*$. To make the notation less heavy-handed, write $\psi_\mu : L_J \cap U \to \mathbb{C}^*$, for $\mathrm{Res}^U_{U\cap L_J}(\psi_\mu)$.

**Lemma 3.1.** *Let $\psi_\mu$ be as in (2.10). Then*

$$\mathrm{Ind}_U^G(\psi_\mu) \cong \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu)).$$

*Proof.* Recall $\mathrm{Ind}_U^G(\psi_\mu) \cong \mathbb{C}Ge_\mu$. On the other hand,

$$\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu) \cong \mathbb{C}L_J e_{\mu J} \quad \text{implies} \quad \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu)) \cong \mathbb{C}Ge_{\mu J}e'_J,$$

where $e_{\mu J}$ is as in (3.3). But $e_{\mu J}e'_J = e_\mu$, so

$$\mathrm{Ind}_U^G(\psi_\mu) \cong \mathbb{C}Ge_\mu \cong \mathbb{C}e_{\mu J}e'_J \cong \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu)).$$

□

**Theorem 3.1.** *The map*

$$\begin{array}{rccc} \theta : & \mathrm{End}_{\mathbb{C}L_J}(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu)) & \longrightarrow & \mathcal{H}_\mu \\ & e_{\mu J} v e_{\mu J} & \mapsto & e_\mu v e_\mu, \quad \textit{for } v \in L_J \cap N_\mu, \end{array}$$

*is an injective algebra homomorphism.*

*Proof.* Since $L_J$ normalizes $U_J$ and $e'_J e_{\mu J} = e_\mu$,

$$e_\mu v e_\mu = e'_J e_{\mu J} v e'_J e_{\mu J} = e'_J e_{\mu J} v e_{\mu J},$$

so the map $\theta$ is given by multiplying $e_{\mu J} v e_{\mu J}$ on the left by $e'_J$. Thus, $\theta$ is well-defined and injective. Because $e'_J$ commutes with $e_{\mu J} v e_{\mu J}$ for $v \in L_J$, $\theta$ is also a homomorphism. □

Write

$$\mathcal{L}_J = \theta(\mathrm{End}_{\mathbb{C}L_J}(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu))) \subseteq \mathcal{H}_\mu \tag{3.4}$$

The $\mathcal{L}_J$ are "parabolic" subalgebras of $\mathcal{H}_\mu$, in that they have a similar role in the representation theory of $\mathcal{H}_\mu$ as parabolic subgroups $P_J$ have in the representation theory of $G$.

### 3.1 Weight space decompositions for $\mathcal{H}_\mu$-modules

An important special case of Theorem 3.1 is when

$$J = J_\mu = \{\alpha_i \text{ simple root } \mid \mu_{\alpha_i} \neq 0\},$$

so that $J_\mu$ is minimal satisfying (3.1). Write $L_\mu = L_{J_\mu}$, $W_\mu = W_{J_\mu}$, etc.

**Corollary 1.** *The algebra $\mathcal{L}_\mu$ is a nonzero commutative subalgebra of $\mathcal{H}_\mu$.*

*Proof.* As a character of $U \cap L_\mu$, $\psi_\mu$ is in general position, so $\mathrm{Ind}_{L_\mu \cap U}^{L_\mu}(\psi_\mu)$ is a Gelfand-Graev module and $\mathcal{L}_\mu$ is a Gelfand-Graev Hecke algebra (see example 2 in Section 2.3). □

Since $\mathcal{L}_\mu$ is commutative, all the irreducible $\mathcal{L}_\mu$-modules are one-dimensional. Let $\hat{\mathcal{L}}_\mu$ be an indexing set for the irreducible modules of $\mathcal{L}_\mu$. Suppose $V$ is an $\mathcal{H}_\mu$-module. Since $\mathcal{L}_\mu \cong \mathrm{End}_{\mathbb{C}L_\mu}(\mathrm{Ind}_{U\cap L_\mu}^{L_\mu}(\psi_\mu))$, $\mathcal{L}_\mu$ is semisimple, and as an $\mathcal{L}_\mu$-module,

$$V \cong \bigoplus_{\gamma \in \hat{\mathcal{L}}_\mu} V_\gamma \qquad \text{where} \quad V_\gamma = \{v \in V \mid xv = \gamma(x)v, x \in \mathcal{L}_\mu\}.$$

If $\gamma \in \hat{\mathcal{L}}_\mu$, then $V_\gamma$ is the $\gamma$-*weight space* of $V$, and $\gamma$ *is a weight of* $V$ if $V_\gamma \neq 0$.

**Examples.**

1. In the Yokonuma algebra $\psi_\mu = \mathbb{1}$, $J_{\mathbb{1}} = \emptyset$ and $\mathcal{L}_{\mathbb{1}} = e_{\mathbb{1}}\mathbb{C}Te_{\mathbb{1}} \cong \mathbb{C}T$.
2. In the Gelfand-Graev Hecke algebra case, $J_\mu = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ and $\mathcal{L}_\mu = \mathcal{H}_\mu$.

**Remark.** Since $\dim(V_\gamma)$ can be greater than one, $\mathcal{L}_\mu$ is not in general a maximal commutative subalgebra of $\mathcal{H}_\mu$.

## 4 Multiplication of basis elements

This section examines the decomposition of products in terms of the natural basis

$$(e_\mu u e_\mu)(e_\mu v e_\mu) = \sum_{v' \in N_\mu} c_{uv}^{v'}\ (e_\mu v' e_\mu).$$

In particular, Theorem 4.1, below, gives a set of braid-like relations (similar to those of the Yokonuma algebra) for manipulating the products, and Corollary 3 gives a recursive formula for computing these products.

### 4.1 Chevalley group relations

The relations governing the interaction between the subgroups $N$, $U$, and $T$ will be critical in describing the Hecke algebra multiplication in the following section. They can all be found in [St, §3].

The subgroup

$$U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q\rangle$$

has generators $\{x_\alpha(t) \mid \alpha \in R^+,\, t \in \mathbb{F}_q\}$, with relations

$$x_\alpha(a)x_\beta(b)x_\alpha(a)^{-1}x_\beta(b)^{-1} = \prod_{\substack{\gamma=i\alpha+j\beta\in R^+\\ i,j\in\mathbb{Z}_{>0}}} x_\gamma(z_{ij}(\alpha,\beta)a^ib^j), \tag{U1}$$

$$x_\alpha(a)x_\alpha(b) = x_\alpha(a+b), \tag{U2}$$

where $z_{ij}(\alpha,\beta) \in \mathbb{Z}$ depends on $i, j, \alpha, \beta$ and a fixed order on the positive roots $R^+$, but not on $a, b \in \mathbb{F}_q$ [St, Lemma 15]. The $z_{ij}(\alpha,\beta)$ have been explicitly computed for various types in [De, St].

The subgroup $N$ has generators $\{w_i(1),\, h_H(t) \mid i = 1,2,\ldots,\ell,\, H \in \mathfrak{h}_{\mathbb{Z}},\, t \in \mathbb{F}_q^*\}$, with relations

$$w_i(1)^2 = h_i(-1), \tag{N1}$$

$$\underbrace{w_i(1)w_j(1)w_i(1)w_j(1)\cdots}_{m_{ij}\text{ terms}} = \underbrace{w_j(1)w_i(1)w_j(1)w_i(1)\cdots}_{m_{ij}\text{ terms}}, \qquad \text{where } (s_is_j)^{m_{ij}} = 1 \text{ in } W, \tag{N2}$$

$$w_i(1)h_H(t) = h_{s_i(H)}(t)w_i(1), \tag{N3}$$

$$h_H(a)h_H(b) = h_H(ab), \tag{N4}$$

$$h_H(a)h_{H'}(b) = h_{H'}(b)h_H(a), \qquad \text{for } H, H' \in \mathfrak{h}, \tag{N5}$$

$$h_H(a)h_{H'}(a) = h_{H+H'}(a), \qquad \text{for } H, H' \in \mathfrak{h}, \tag{N6}$$

$$h_{H_1}(t_1)h_{H_2}(t_2)\ldots h_{H_k}(t_k) = 1, \qquad \text{if } t_1^{\lambda_j(H_1)}\cdots t_k^{\lambda_j(H_k)} = 1 \text{ for all } 1 \le j \le r, \tag{N7}$$

where $\lambda_j : \mathfrak{h} \to \mathbb{C}$ depends on $V$ as in (2.4).

The double-coset decomposition of $G$ (2.13) implies $G = \langle U, N\rangle$. Thus, $G$ is generated by $\{x_\alpha(a), w_i(1), h_H(b) \mid \alpha \in R^+, a \in \mathbb{F}_q, i = 1,2,\ldots,\ell, H \in \mathfrak{h}_{\mathbb{Z}}, b \in \mathbb{F}_q^*\}$ with relations (U1)-(N7) and

$$w_i(1)x_\alpha(t)w_i(1)^{-1} = x_{s_i(\alpha)}(c_{i\alpha}t), \qquad \text{for } \alpha \ne \alpha_i, \text{ where } c_{i\alpha} = \pm 1, \tag{UN1}$$

$$hx_\alpha(b)h^{-1} = x_\alpha(\alpha(h)b), \qquad \text{for } h \in T, \tag{UN2}$$

$$w_i(1)x_i(t)w_i(1) = x_i(-t^{-1})h_i(-t^{-1})w_i(1)x_i(-t^{-1}), \qquad \text{where } x_i(t) = x_{\alpha_i}(t) \text{ and } t \ne 0, \tag{UN3}$$

where for $\alpha \in R$ and $h_H(t) \in T$,

$$\alpha(h_H(t)) = t^{\alpha(H)}. \tag{4.1}$$

Note that relation (UN3) is not conjugation by $w_i(1)$,

Fix a $\psi_\mu : U \to \mathbb{C}^*$ as in (2.10). For $k \in \mathbb{F}_q$, let

$$e_\alpha(k) = \frac{1}{q}\sum_{t\in\mathbb{F}_q} \psi(-\mu_\alpha kt)x_\alpha(t) \quad \text{with the convention} \quad e_\alpha = e_\alpha(1). \tag{4.2}$$

Note that for any given ordering of the positive roots, the decomposition

$$U = \prod_{\alpha\in R^+} U_\alpha \qquad \text{implies} \qquad e_\mu = \prod_{\alpha\in R^+} e_\alpha. \tag{4.3}$$

In particular, given any $\alpha \in R^+$, we may choose the ordering of the positive roots to have $e_\alpha$ appear either first or last. Therefore, since $e_\alpha$ is an idempotent,

$$e_\mu e_\alpha = e_\mu = e_\alpha e_\mu. \tag{4.4}$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ with $r$ minimal, then let

$$R_w = \{\alpha \in R^+ \mid w(\alpha) \in R^-\} = \{\alpha_{i_r}, s_{i_r}(\alpha_{i_{r-1}}), \ldots, s_{i_r}s_{i_{r-1}}\cdots s_{i_2}(\alpha_{i_1})\}, \tag{4.5}$$

where the second equality is from [Bo, VI.1, Corollary 2 of Proposition 17].

**Lemma 4.1.** *Let $v \in N$, $w = \pi(v)$ (with $\pi : N \to W$ as in (2.6)), and for $\alpha \in R^+$, let $vx_\alpha(t)v^{-1} = x_{w\alpha}(c_{v\alpha}t)$, with $c_{v\alpha} = \pm 1$ as in (UN1). Then*

$$\begin{aligned}
ve_\alpha(k)v^{-1} &= e_{w\alpha}(\mu_\alpha\mu_{w\alpha}^{-1}c_{v\alpha}k), && \text{if } \alpha \notin R_w, && \text{(E1)}\\
ve_\alpha v^{-1} &= e_{w\alpha}, && \text{if } \alpha \notin R_w, v \in N_\mu, && \text{(E2)}\\
he_\alpha(k)h^{-1} &= e_\alpha(k\alpha(h)^{-1}), && \text{for } h \in T, && \text{(E3)}\\
e_\mu x_\alpha(t) &= \psi(\mu_\alpha t)e_\mu = x_\alpha(t)e_\mu, && \text{for } \alpha \in R^+. && \text{(E4)}
\end{aligned}$$

*Proof.* (E1) Using relation (UN1),

$$\begin{aligned}
we_\alpha(k)w^{-1} &= \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)wx_\alpha(t)w^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)x_{w\alpha}(c_{v\alpha}t)\\
&= \frac{1}{q}\sum_{t'\in\mathbb{F}_q}\psi(-\mu_\alpha c_{v\alpha}kt')x_{w\alpha}(t') = e_{w\alpha}(\mu_\alpha\mu_{w\alpha}^{-1}c_{v\alpha}k).
\end{aligned}$$

(E2) Suppose $\alpha \notin R_w$. Since $v \in N_\mu$,

$$\begin{aligned}
\psi(\mu_\alpha t) = \psi_\mu(x_\alpha(t)) = \psi_\mu(vx_\alpha(t)v^{-1}) &= \psi_\mu(x_{w\alpha}(kt)) && \text{(by (UN1))}\\
&= \psi(\mu_{w\alpha}kt), \quad \text{for some } k \in \mathbb{Z}_{\neq 0}.
\end{aligned}$$

In particular, since $\psi$ is nontrivial, $\mu_\alpha = k\mu_{w\alpha}$. Thus,

$$ve_\alpha v^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha t)x_{w\alpha}(kt) = \frac{1}{q}\sum_{t'\in\mathbb{F}_q}\psi(-\mu_\alpha k^{-1}t')x_{w\alpha}(t') = e_{w\alpha}.$$

(E3) Since $hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t)$,

$$he_\alpha(k)h^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)x_\alpha(\alpha(h)t) = \sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt\alpha(h)^{-1})x_\alpha(t) = e_\alpha(k\alpha(h)^{-1}).$$

(E4) The element $e_\alpha$ is the minimal central idempotent of $\mathbb{C}U_\alpha$ that corresponds to the character $x_\alpha(t) \mapsto \psi(\mu_\alpha t)$. Therefore, by (4.4), $e_\mu x_\alpha(t) = e_\mu e_\alpha x_\alpha(t) = \psi(\mu_\alpha t)e_\mu$. □

## 4.2 Local Hecke algebra relations

Let $u = u_1u_2\cdots u_ru_T \in N$ according to a minimal expression $s_{i_1}s_{i_2}\ldots s_{i_r} \in W$ (see (2.8)). For $1 \le k \le r$ define constants $c_k = \pm 1$ and roots $\beta_k \in R^+$ by the equation

$$x_{\beta_k}(c_kt) = (u_{k+1}\cdots u_r)^{-1}x_{\alpha_{i_k}}(t)(u_{k+1}\cdots u_r). \tag{4.6}$$

Note that $R_{\pi(u)} = \{\beta_1, \beta_2, \ldots, \beta_r\}$ (see (4.5)). Define $f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r]$ by

$$f_u = -\frac{\mu_{\beta_1}c_1}{\beta_1(u_T)}y_1 - \frac{\mu_{\beta_2}c_2}{\beta_2(u_T)}y_2 - \cdots - \frac{\mu_{\beta_r}c_r}{\beta_r(u_T)}y_r, \tag{4.7}$$

and for $k = 1, 2, \ldots, r$, and write

$$u_k(t) = w_{i_k}(1)x_{i_k}(t). \tag{4.8}$$

In the following theorem we evaluate polynomials $f \in \mathbb{F}_q[y_1, \ldots, y_r]$ at points in $t = (t_1, \ldots, t_r) \in \mathbb{F}_q^r$ by $f(t) = f(t_1, \ldots, t_r)$, where $y_j(t) = t_j$ for $1 \le j \le r$.

**Theorem 4.1.** *Let $u = u_1u_2\cdots u_ru_T, v = v_1v_2\cdots v_sv_T \in N_\mu$ according to minimal expressions $s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ and $s_{j_1}s_{j_2}\cdots s_{j_s} \in W$, respectively, as in (2.8). Then*

*(a)*

$$(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f_u)(t)\ e_\mu\big(u_1(t_1)u_2(t_2)\cdots u_r(t_r)\big)\big(v_1v_2\cdots v_s\big)he_\mu,$$

*where $h = v_Tv^{-1}u_Tv \in T$.*

*(b) The following local relations suffice to compute the product $(e_\mu u e_\mu)(e_\mu v e_\mu)$.*

$$\sum_{t\in\mathbb{F}_q}(\psi\circ f)(t)\big(w_i(1)x_i(t)\big)w_i(1) = (\psi\circ f)(0)h_i(-1) + \sum_{t\in\mathbb{F}_q^*}(\psi\circ f)(-t^{-1})x_i(t)h_i(t)w_i(1)x_i(t), \quad (\mathcal{H}1)$$

$$w_i(1)x_\alpha(t) = x_{s_i(\alpha)}(c_{i\alpha}t)w_i(1), \quad (\mathcal{H}2)$$

$$x_\alpha(t)h = hx_\alpha(\alpha(h)^{-1}t), \quad (\mathcal{H}3)$$

$$e_\mu x_\alpha(t) = \psi(\mu_\alpha t)e_\mu = x_\alpha(t)e_\mu, \quad (\mathcal{H}4)$$

$$(\psi\circ f)(t)(\psi\circ g)(t) = (\psi\circ(f+g))(t), \quad (\mathcal{H}5)$$

$$h_\alpha(t)w_i(1) = w_i(1)h_{s_i(\alpha)}(t), \quad (\mathcal{H}6)$$

$$\big(w_i(1)x_i(a)\big)x_\alpha(b) = \prod_{\substack{\gamma=m\alpha_i+n\alpha\in R^+\\ m\geq 0,n>0}} x_{s_i\gamma}(c_{i\gamma}z_{mn}(\alpha_i,\alpha)a^mb^n)\big(w_i(1)x_i(a)\big), \quad \textit{for } \alpha\neq\alpha_i, (\mathcal{H}7)$$

$$\big(w_i(1)x_i(a)\big)x_i(b) = \big(w_i(1)x_i(a+b)\big), \quad (\mathcal{H}8)$$

$$h_\alpha(a)h_\alpha(b) = h_\alpha(ab), \quad (\mathcal{H}9)$$

$$h_\alpha(a)h_\beta(b) = h_\beta(b)h_\alpha(a) \quad (\mathcal{H}10)$$

$$w_i(1)^2 = h_i(-1) \quad (\mathcal{H}11)$$

$$\underbrace{w_i(1)w_j(1)w_i(1)w_j(1)\cdots}_{m_{ij}\ terms} = \underbrace{w_j(1)w_i(1)w_j(1)w_i(1)\cdots}_{m_{ij}\ terms}, \quad (\mathcal{H}12)$$

*where $f, g \in \mathbb{F}_q[y_1^{\pm1},\ldots,y_r^{\pm1}]$, $t\in\mathbb{F}_q^r$, $\alpha,\beta\in R^+$, $1\leq i\leq \ell$, $z_{0,1}(\alpha_i,\alpha) = 1$, and $m_{ij}$ is the order of $s_is_j$ in $W$.*

*Proof.* **(a)** Order the positive roots so that by (4.3)

$$\begin{aligned}
e_\mu u e_\mu v e_\mu &= e_\mu u\Big(\prod_{\alpha\notin R_{\pi(u)}} e_\alpha\Big)e_{\beta_1}e_{\beta_2}\cdots e_{\beta_r}ve_\mu && \text{(definition of } \beta_k)\\
&= e_\mu\Big(\prod_{\alpha\notin R_{\pi(u)}} e_{\pi(u)\alpha}\Big)ue_{\beta_1}e_{\beta_2}\cdots e_{\beta_r}ve_\mu && \text{(Lemma 4.1,E2)}\\
&= e_\mu u e_{\beta_1}e_{\beta_2}\cdots e_{\beta_r}ve_\mu && \text{(Lemma 4.1, E4)}\\
&= e_\mu u_1u_2\cdots u_ru_Te_{\beta_1}e_{\beta_2}\cdots e_{\beta_r}ve_\mu\\
&= e_\mu u_1u_2\cdots u_re_{\beta_1}(\tfrac{1}{\beta_1(u_T)})e_{\beta_2}(\tfrac{1}{\beta_2(u_T)})\cdots e_{\beta_r}(\tfrac{1}{\beta_r(u_T)})u_Tve_\mu && \text{(Lemma 4.1,E3)}\\
&= e_\mu u_1e_{\alpha_{i_1}}(\tfrac{\mu_{\beta_1}c_1}{\mu_{\alpha_{i_1}}\beta_1(u_T)})u_2e_{\alpha_{i_2}}(\tfrac{\mu_{\beta_2}c_2}{\mu_{\alpha_{i_2}}\beta_2(u_T)})\cdots u_re_{\alpha_{i_r}}(\tfrac{\mu_{\beta_r}c_r}{\mu_{\alpha_{i_r}}\beta_r(u_T)})u_Tve_\mu && \text{((4.6), Lemma 4.1,E1)}\\
&= e_\mu u_1e_{\alpha_{i_1}}(\tfrac{\mu_{\beta_1}c_1}{\mu_{\alpha_{i_1}}\beta_1(u_T)})u_2e_{\alpha_{i_2}}(\tfrac{\mu_{\beta_2}c_2}{\mu_{\alpha_{i_2}}\beta_2(u_T)})\cdots u_re_{\alpha_{i_r}}(\tfrac{\mu_{\beta_r}c_r}{\mu_{\alpha_{i_r}}\beta_r(u_T)})v_1\cdots v_sv_Tv^{-1}u_Tve_\mu\\
&= \frac{e_\mu}{q^r}\sum_{t_1,\ldots,t_r\in\mathbb{F}_q}\psi(-\tfrac{\mu_{\beta_1}c_1t_1}{\beta_1(u_T)})u_1(t_1)\cdots\psi(-\tfrac{\mu_{\beta_r}c_rt_r}{\beta_r(u_T)})u_r(t_r)v_1\cdots v_she_\mu && \text{(definition of } e_\alpha,\ u_\alpha(t))
\end{aligned}$$

$$= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t) e_\mu u_1(t_1) \cdots u_r(t_r) v_1 \cdots v_s h e_\mu, \qquad \text{(by } (\mathcal{H}5))$$

where $h = v_T v^{-1} u_T v \in T$, as desired.

**(b)** First, note that these relations are in fact correct (though not necessarily sufficient): $(\mathcal{H}1)$ comes from (UN3); $(\mathcal{H}2)$ comes from (UN1); $(\mathcal{H}3)$ comes from (UN2); $(\mathcal{H}4)$ is (E4); $(\mathcal{H}5)$ comes from the multiplicativity of $\psi$; $(\mathcal{H}6)$ comes from (N3); $(\mathcal{H}7)$ comes from (U1) and (UN1); $(\mathcal{H}8)$ comes from (U2); $(\mathcal{H}9)$ and $(\mathcal{H}10)$ are (N4) and (N5); and $(\mathcal{H}11)$ and $(\mathcal{H}12)$ are (N1) and (N2). It therefore remains to show sufficiency.

By (a) we may write

$$(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_r(t_r) v_1 \cdots v_s h e_\mu$$

for some $f \in \mathbb{F}_q[y_1, \ldots, y_r]$ and $h \in T$. Say $t_k$ is *resolved* if the only parts of the summands depending on $t_k$ are $(\psi \circ f)$ and $h$. The product is *reduced* when all the $t_k$ are resolved. I will show how to resolve $t_r$ and the result will follow by induction.

Use relation $(\mathcal{H}2)$ to define the constant $d$ and the root $\gamma \in R$ by

$$(v_1 v_2 \cdots v_s)^{-1} x_{\alpha_{i_r}}(t)(v_1 v_2 \cdots v_s) = x_\gamma(dt) \qquad (\text{where } \ell(\pi(v)) = s). \tag{4.9}$$

Note that $\gamma = \pi(v)^{-1}(\alpha_{i_r})$ and $d = \pm 1$. There are two possible situations:

**Case 1.** $\gamma \in R^+$,

**Case 2.** $\gamma \in R^-$.

In Case 1,

$$\begin{aligned}
(e_\mu u e_\mu)(e_\mu v e_\mu) &= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_r \underrightarrow{x_{i_r}(t_r)} v_1 \cdots v_s h e_\mu && \text{(by (a))} \\
&= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r v_1 \cdots v_s \underrightarrow{x_\gamma(dt_r)} h e_\mu && \text{(by } (\mathcal{H}2)) \\
&= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r v_1 \cdots v_s h \underline{x_\gamma(d\gamma(h)^{-1} t_r) e_\mu} && \text{(by } (\mathcal{H}3)) \\
&= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r v_1 \cdots v_s h \underline{\psi(\mu_\gamma d\gamma(h)^{-1} t_r)} e_\mu && \text{(by } (\mathcal{H}4)) \\
&= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r v_1 \cdots v_s h \underleftarrow{(\psi \circ \mu_\gamma d\gamma(h)^{-1} y_r)(t)} e_\mu \\
&= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ g)(t) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r v_1 \cdots v_s h e_\mu, && \text{(by } (\mathcal{H}5))
\end{aligned}$$

where $g = f + \mu_\gamma d\gamma(h)^{-1} y_r$. We have resolved $t_r$ in Case 1. Furthermore, since $\gamma \in R^+$, $v_1' v_2' \cdots v_{s+1}' = u_r v_1 v_2 \cdots v_s$ still corresponds to a minimal expression in $W$.

In Case 2, $\gamma \in R^-$, so we can no longer move $x_{i_r}(t_r)$ past the $v_j$. Instead,

$$
\begin{aligned}
&(e_\mu u e_\mu)(e_\mu v e_\mu)\\
&= \frac{e_\mu}{q^r} \sum_{t \in \mathbb{F}_q^r} \underline{(\psi \circ f)(t)} u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r x_{i_r}(t_r) u_r u_r^{-1} v_1 \cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \sum_{t_r \in \mathbb{F}_q} (\psi \circ f)(t', t_r) \underline{u_r x_{i_r}(t_r) u_r} u_r^{-1} v_1 \cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) (\psi \circ f)(t', 0) \underrightarrow{h_{i_r}(-1)} u_r^{-1} v_1 \cdots v_s h e_\mu && \text{(by } (\mathcal{H}1))\\
&\quad + \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \sum_{t_r \in \mathbb{F}_q^*} (\psi \circ f)(t', -t_r^{-1}) x_{i_r}(t_r) h_{i_r}(t_r) \underline{u_r x_{i_r}(t_r) u_r^{-1}} v_1 \cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \underleftarrow{(\psi \circ f)(t', 0)} u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu && \text{(by } (\mathcal{H}6, \mathcal{H}2))\\
&\quad + \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \sum_{t_r \in \mathbb{F}_q^*} (\psi \circ f)(t', -t_r^{-1}) x_{i_r}(t_r) h_{i_r}(t_r) \underrightarrow{x_{-\alpha_{i_r}}(-t_r)} v_1 \cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} (\psi \circ f)(t', 0) u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu\\
&\quad + \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \sum_{t_r \in \mathbb{F}_q^*} (\psi \circ g)(t', -t_r^{-1}) x_{i_r}(t_r) \underrightarrow{h_{i_r}(t_r)} v_1 \cdots v_s h e_\mu, && \text{(by } (\mathcal{H}3, \mathcal{H}4, \mathcal{H}5))
\end{aligned}
$$

where $g = f + \mu_{-\gamma} d(-\gamma(h))^{-1} y_r^{-1}$ (same as in the analogous steps in Case 1).

$$
\begin{aligned}
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} (\psi \circ f)(t', 0) u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu\\
&\quad + \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} u_1(t_1) \cdots u_{r-1}(t_{r-1}) \sum_{t_r \in \mathbb{F}_q^*} \underleftarrow{(\psi \circ g)(t', -t_r^{-1}) x_{i_r}(t_r)} v_1 \cdots v_s h_{-\gamma}(t_r^{-1}) h e_\mu, && \text{(by } (\mathcal{H}6))\\
&= \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} (\psi \circ f)(t', 0) u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu && \text{(by } (\mathcal{H}7, \mathcal{H}8))\\
&\quad + \frac{e_\mu}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1}\\ t_r \in \mathbb{F}_q^*}} (\psi \circ \varphi(g))(t', -t_r^{-1}) \left( \prod_{\beta \in R^+} \underleftarrow{x_\beta(a_\beta(t', t_r))} \right) u_1(t_1) \cdots u_{r-1}(t_{r-1}) v_1 \cdots v_s h' e_\mu
\end{aligned}
$$

where $\varphi : \mathbb{F}_q[y_1, \ldots, y_r] \to \mathbb{F}_q[y_1, \ldots, y_r]$ catalogues the substitutions to $g$ due to $(\mathcal{H}8)$ and $(\mathcal{H}2)$, the $a_\beta(y_1, y_2, \ldots, y_r) \in \mathbb{F}_q[y_1, \ldots, y_r]$ are determined by repeated applications of $(\mathcal{H}7)$ and $(\mathcal{H}8)$, and $h' = h_{-\gamma}(t_r^{-1}) h \in T$.

$$
\begin{aligned}
&= \frac{1}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} (\psi \circ f)(t', 0) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu && \text{(by } (\mathcal{H}4))\\
&\quad + \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1}\\ t_r \in \mathbb{F}_q^*}} (\psi \circ \varphi(g))(t', -t_r^{-1}) e_\mu \left( \prod_{\beta \in R^+} \underleftarrow{\psi(\mu_\beta a_\beta(t', t_r))} \right) u_1(t_1) \cdots u_{r-1}(t_{r-1}) v_1 \cdots v_s h' e_\mu
\end{aligned}
$$

$$= \frac{1}{q^r} \sum_{t' \in \mathbb{F}_q^{r-1}} (\psi \circ f)(t', 0) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) u_r^{-1} v_1 \cdots v_s h_{-\gamma}(-1) h e_\mu$$
$$+ \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r \in \mathbb{F}_q^*}} (\psi \circ g_2)(t', -t_r^{-1}) e_\mu u_1(t_1) \cdots u_{r-1}(t_{r-1}) v_1 \cdots v_s h' e_\mu, \qquad \text{(by } (\mathcal{H}5))$$

where $g_2 = \varphi(g) + \sum_{\beta \in R^+} \mu_\beta a_\beta(y_1, \ldots, y_{r-1}, -y_r^{-1})$. In the first sum, use $(\mathcal{H}11)$ and $(\mathcal{H}12)$ to reduce $v'_1 \cdots v'_{s-1} = u_r^{-1} v_1 \cdots v_s$ into an expression that corresponds to a minimal expression in $W$. Use $(\mathcal{H}9)$ and $(\mathcal{H}10)$ to simplify the expressions $h', h_{-\gamma}(-1)h \in T$. Now $t_r$ is resolved for Case 2, as desired. ◻

**Corollary 2 (Resolving $t_k$).** *Let $u = u_1 u_2 \cdots u_k \in N$ according to a minimal expression $s_{i_1} s_{i_2} \cdots s_{i_k} \in W$ (with $u_T = 1$). Suppose $v \in N$ and $f \in \mathbb{F}_q[y_1, y_2, \ldots, y_k]$. Define $\gamma \in R$ and $d \in \mathbb{C}$ by the equation $v^{-1} x_{i_k}(t) v = x_\gamma(dt)$. Then*

**Case 1.** *If $\ell(\pi(u_k v)) > \ell(\pi(v))$, then*

$$\sum_{t \in \mathbb{F}_q^k} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_k(t_k) v e_\mu = \sum_{t \in \mathbb{F}_q^k} (\psi \circ (\underline{f + \mu_\gamma d y_k}))(t) e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}) \underline{u_k v} e_\mu.$$

**Case 2.** *If $\ell(\pi(u_k v)) < \ell(\pi(v))$, then*

$$\sum_{t \in \mathbb{F}_q^k} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_k(t_k) v e_\mu = \sum_{\substack{t \in \mathbb{F}_q^k, \\ \underline{t_k = 0}}} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}) \underline{u_k v} e_\mu$$
$$+ \sum_{\substack{t \in \mathbb{F}_q^k \\ \underline{t_k \in \mathbb{F}_q^*}}} (\psi \circ (\underline{\varphi_k(f) + \mu_{-\gamma} d y_k^{-1}}))(t) e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}) \underline{h_{i_k}(-t_k) v} e_\mu,$$

*where $\varphi_k : \mathbb{F}_q[y_1^{\pm 1}, \ldots, y_k^{\pm 1}] \to \mathbb{F}_q[y_1^{\pm 1}, \ldots, y_k^{\pm 1}]$ is given by*

$$\sum_{\substack{t \in \mathbb{F}_q^k \\ t_k \in \mathbb{F}_q^*}} (\psi \circ f)(t) e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}) \underleftarrow{x_{i_k}(-t_k^{-1})} = \sum_{\substack{t \in \mathbb{F}_q^k \\ t_k \in \mathbb{F}_q^*}} (\psi \circ \underline{\varphi_k(f)})(t) e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}).$$

*Proof.* This Corollary puts $v$ in the place of $u_T v$ in the proof of Theorem 4.1, (b), and summarizes the steps taken in Case 1 and Case 2. The only slight adjustments are in Case 2: note that $u_k v = h_{i_k}(-1) u_k^{-1} v$ in the first summand, and there is a renormalization of $t_k$ in the second summand. ◻

### 4.3 Global Hecke algebra relations

Fix a decomposition $u = u_1 u_2 \cdots u_r u_T \in N_\mu$ according a minimal expression $s_{i_1} s_{i_2} \cdots s_{i_r} \in W$ (see (2.8)). Suppose $v' \in N_\mu$ and let $v = u_T v'$.

For $0 \le k \le r$, let $\tau = (\tau_1, \tau_2, \ldots, \tau_{r-k})$ be such that $\tau_i \in \{+0, -0, 1\}$, where $+0$, $-0$, and $1$ are symbols. If $\tau$ has $r - k$ elements, then the *colength* of $\tau$ is $\ell^\vee(\tau) = k$. For example, if $r = 10$ and $\tau = (-0, 1, +0, +0, 1, 1)$, then $\ell^\vee(\tau) = 4$. For $i \in \{+0, -0, 1\}$, let

$$(i, \tau) = (i, \tau_1, \tau_2, \cdots, \tau_{r-k}).$$

By convention, if $\ell^\vee(\tau) = r$, then $\tau = \emptyset$.

Suppose $\ell^\vee(\tau) = k$. Define

$$\Xi^\tau(u,v) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^\tau} (\psi \circ f^\tau)(t) e_\mu u_1(t_1) \cdots u_k(t_k) v^\tau(t) e_\mu, \tag{4.10}$$

where

$$\mathbb{F}_q^\tau = \left\{ t \in \mathbb{F}_q^r \;\middle|\; \text{for } k < i \le r, \begin{array}{l} \text{if } \tau_{i-k} = +0, \text{ then } t_i \in \mathbb{F}_q, \\ \text{if } \tau_{i-k} = -0, \text{ then } t_i = 0, \\ \text{if } \tau_{i-k} = 1, \text{ then } t_i \in \mathbb{F}_q^*. \end{array} \right\}; \tag{4.11}$$

$$v^\tau(t) = h_{i_{k+1}}(-t_{k+1})^{\tau_1} u_{k+1}^{1-\tau_1} \cdots h_{i_r}(-t_r)^{\tau_{r-k}} u_r^{1-\tau_{r-k}} v, \tag{4.12}$$

with $+0 = -0 = 0 \in \mathbb{Z}$, $1 = 1 \in \mathbb{Z}$ in (4.12); and $f^\tau$ is defined recursively by

$$f^\emptyset = f_u = -\frac{\mu_{\beta_1} c_1}{\beta_1(u_T)} y_1 - \frac{\mu_{\beta_2} c_2}{\beta_2(u_T)} y_2 - \cdots - \frac{\mu_{\beta_r} c_r}{\beta_r(u_T)} y_r, \quad \text{(as in (4.7))}, \tag{4.13}$$

$$f^{(i,\tau)} = \begin{cases} f^\tau + \mu_{\gamma_\tau} d_\tau y_k, & \text{if } i = +0, \\ f^\tau, & \text{if } i = -0, \\ \varphi_k(f^\tau) + \mu_{-\gamma_\tau} d_\tau y_k^{-1}, & \text{if } i = 1, \end{cases} \tag{4.14}$$

where $(v^\tau)^{-1} x_{\alpha_{i_k}}(t) v^\tau = x_{\gamma_\tau}(d_\tau t)$ and the map $\varphi_k$ is as in Corollary 2, Case 2.

**Remarks**.

1. By (4.10) and Theorem 4.1 (a), $\Xi^\emptyset(u,v) = (e_\mu u e_\mu)(e_\mu v' e_\mu)$ (recall, $v = u_T v'$).
2. If $\ell^\vee(\tau) = 0$ so that $\tau$ is a string of length $r$, then
   (a) $\Xi^\tau(u,v) = \dfrac{1}{q^r} \sum_{t \in \mathbb{F}_q^\tau} (\psi \circ f^\tau)(t) e_\mu v^\tau(t) e_\mu$ has no remaining factors of the form $u_k(t_k)$,
   (b) $\Xi^\tau(u,v) = 0$ unless $v^\tau(t) \in N_\mu$ for some $t \in \mathbb{F}_q$.

The following corollary gives relations for expanding $\Xi^\tau(u,v)$ (beginning with $\Xi^\emptyset(u,v)$) as a sum of terms of the form $\Xi^{\tau'}$ with $\ell^\vee(\tau') = \ell^\vee(\tau) - 1$. When each term has colength 0 (length $r$), then the product $(e_\mu u e_\mu)(e_\mu v' e_\mu)$ is decomposed in terms of the basis elements of $\mathcal{H}_\mu$.

In summary, while we compute $f^\tau$ recursively by *removing* elements from $\tau$, we compute the product $(e_\mu u e_\mu)(e_\mu v' e_\mu)$ by progressively *adding* elements to $\tau$.

**Corollary 3 (The Global Alternative).** *Let $u, v' \in N_\mu$ such that $u = u_1 u_2 \cdots u_r u_T$ decomposes according to a minimal expression in $W$. Let $v = u_T v'$. Then*

*(a)* $(e_\mu u e_\mu)(e_\mu v' e_\mu) = \Xi^\emptyset(u,v)$,

*(b) If $\ell^\vee(\tau) = k$, then*

$$\Xi^\tau(u,v) = \begin{cases} \Xi^{(+0,\tau)}(u,v), & \textit{if } \ell(\pi(u_k v^\tau)) > \ell(\pi(v^\tau)), \\ \Xi^{(-0,\tau)}(u,v) + \Xi^{(1,\tau)}(u,v), & \textit{if } \ell(\pi(u_k v^\tau)) < \ell(\pi(v^\tau)). \end{cases}$$

*Proof.* (a) follows from Remark 1.
(b) Suppose $\ell^\vee(\tau) = k$. Note that

$$\begin{aligned}\Xi^\tau(u,v) &= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^\tau}(\psi\circ f^\tau)(t)e_\mu u_1(t_1)\cdots u_k(t_k)v^\tau e_\mu \\ &= \frac{1}{q^r}\sum_{t''\in(\mathbb{F}_q^{r-k})^\tau}\sum_{t'\in\mathbb{F}_q^k}(\psi\circ f^\tau)(t',t'')e_\mu u_1(t_1)\cdots u_k(t_k)v^\tau e_\mu\end{aligned}$$

where $(\mathbb{F}_q^{r-k})^\tau = \{(t_{k+1},\dots,t_r)\in\mathbb{F}_q^{r-k} \mid \text{restrictions according to } \tau\}$ (as in (4.11)). Apply Corollary 2 to the inside sum with $f := f^\tau$, $v := v^\tau$. Note that the Corollary relations imply

$$\{t'\in\mathbb{F}_q^k\} \quad \text{becomes} \quad \begin{cases}\{t'\in\mathbb{F}_q^k\}, & \text{if in Case 1,}\\ \{t'\in\mathbb{F}_q^k \mid t_k = 0\}, & \text{if in Case 2, first sum,}\\ \{t'\in\mathbb{F}_q^k \mid t_k\in\mathbb{F}_q^*\}, & \text{if in Case 2, second sum,}\end{cases}$$

$$f^\tau \quad \text{becomes} \quad \begin{cases} f^{(+0,\tau)}, & \text{if in Case 1,}\\ f^{(-0,\tau)}, & \text{if in Case 2, first sum,}\\ f^{(1,\tau)}, & \text{if in Case 2, second sum.}\end{cases}$$

$$v^\tau \quad \text{becomes} \quad \begin{cases} v^{(+0,\tau)}, & \text{if in Case 1,}\\ v^{(-0,\tau)}, & \text{if in Case 2, first sum,}\\ v^{(1,\tau)}, & \text{if in Case 2, second sum.}\end{cases}$$

Thus,

$$\Xi^\tau(u,v) = \begin{cases}\Xi^{(+0,\tau)}(u,v), & \text{if Case 1,}\\ \Xi^{(-0,\tau)}(u,v)+\Xi^{(1,\tau)}(u,v), & \text{if Case 2,}\end{cases}$$

as desired. □

## 5 The case $G = GL_n(\mathbb{F}_q)$

Let $G = GL_n(\mathbb{F}_q)$ be the general linear group over the finite field $\mathbb{F}_q$ with $q$ elements. This section uses braid-like diagrams to analyze multiplication in unipotent Hecke algebras. The structure of this section is as follows.

**5.1** describes the braid-like diagrams of this paper, and how the Chevalley relations translate into diagram relations.

**5.2** reviews unipotent Hecke algebras for $GL_n(\mathbb{F}_q)$ in this context, and shows how to identify the diagrams of unipotent Hecke algebra basis elements.

**5.3** uses a 3 step process to multiply basis elements using the visual cues of the diagrams.

**5.4** summarizes a complete algorithm for multiplying basis elements, and illustrates the process with a nontrivial example.

Define subgroups

$$\begin{gathered} T = \left\{\begin{matrix}\text{diagonal}\\ \text{matrices}\end{matrix}\right\}, \quad N = \left\{\begin{matrix}\text{monomial}\\ \text{matrices}\end{matrix}\right\}, \\ W = \left\{\begin{matrix}\text{permutation}\\ \text{matrices}\end{matrix}\right\}, \quad \text{and} \quad U = \left\{\begin{pmatrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\},\end{gathered} \tag{5.1}$$

where a monomial matrix is a matrix with exactly one nonzero entry in each row and column.

Let $x_{ij}(t) \in U$ be the matrix with $t$ in position $(i,j)$, ones on the diagonal and zeroes elsewhere; write $x_i(t) = x_{i,i+1}(t)$. Let $h_{\varepsilon_i}(t) \in T$ denote the diagonal matrix with $t$ in the $i$th slot and ones elsewhere, and let $s_i \in W \subseteq N$ be the identity matrix with the $i$th and $(i+1)$st columns interchanged. That is,

$$\begin{gathered} x_i(t) = Id_{i-1} \oplus \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \oplus Id_{n-i-1}, \quad h_{\varepsilon_i}(t) = Id_{i-1} \oplus (t) \oplus Id_{n-i}, \\ s_i = Id_{i-1} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus Id_{n-i-1}, \end{gathered} \tag{5.2}$$

where $Id_k$ is the $k \times k$ identity matrix. Then

$$\begin{gathered} W = \langle s_1, s_2, \dots s_{n-1} \rangle, \qquad T = \langle h_{\varepsilon_i}(t) \mid 1 \le i \le n, t \in \mathbb{F}_q^* \rangle, \qquad N = WT, \\ U = \langle x_{ij}(t) \mid 1 \le i < j \le n, t \in \mathbb{F}_q \rangle, \qquad G = \langle U, W, T \rangle. \end{gathered} \tag{5.3}$$

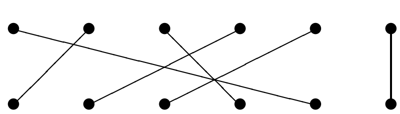
The Chevalley group relations for $G$ are (see also Section 4.1)

$$\begin{aligned} x_{ij}(a) x_{rs}(b) &= x_{rs}(b) x_{ij}(a) x_{is}(\delta_{jr} ab) x_{rj}(-\delta_{is} ab), && (i,j) \neq (r,s), && \text{(U1)} \\ x_{ij}(a) x_{ij}(b) &= x_{ij}(a+b), && && \text{(U2)} \\ s_i^2 &= 1, && && \text{(N1)} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{and} \quad s_i s_j = s_j s_i, && |i-j| > 1, && \text{(N2)} \\ s_i h_{\varepsilon_j}(a) &= h_{\varepsilon_{s_i(j)}}(a) s_i, && && \text{(N3)} \\ h_{\varepsilon_i}(b) h_{\varepsilon_i}(a) &= h_{\varepsilon_i}(ab), && && \text{(N4)} \\ h_{\varepsilon_j}(b) h_{\varepsilon_i}(a) &= h_{\varepsilon_i}(a) h_{\varepsilon_j}(b), && && \text{(N5)} \\ s_r x_{ij}(t) &= x_{s_r(i) s_r(j)}(t) s_r, && && \text{(UN1)} \\ x_{ij}(a) h_{\varepsilon_r}(t) &= h_{\varepsilon_r}(t) x_{ij}(t^{-\delta_{ri}} t^{\delta_{rj}} a), && && \text{(UN2)} \\ s_i x_i(t) s_i &= x_i(t^{-1}) s_i x_i(-t) h_{\varepsilon_i}(t) h_{\varepsilon_{i+1}}(-t^{-1}), && t \neq 0, && \text{(UN3)} \end{aligned}$$

where $\delta_{ij}$ is the Kronecker delta.

## 5.1 A pictorial version of $GL_n(\mathbb{F}_q)$

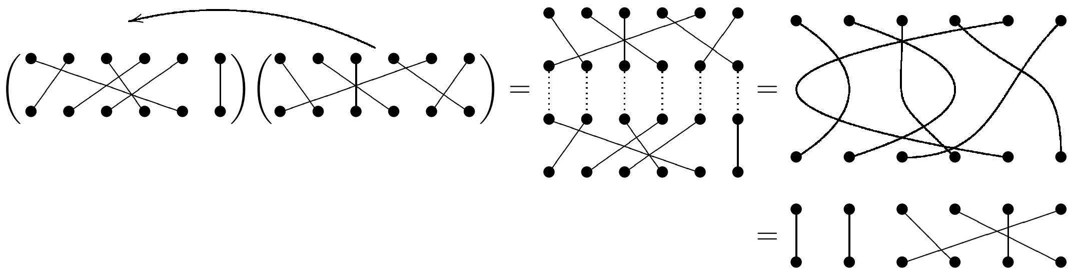
For the results that follow, it will be useful to view elements of $\mathbb{C}G$ as braid-like diagrams instead of matrices. The basic idea is to depict an $n \times n$ permutation matrix $w$ as two rows of $n$ vertices each, with an edge (called a strand) from the $i$th top vertex to the $j$th bottom vertex if $w(i) = j$. For example,



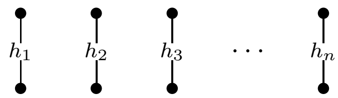
corresponds to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix multiplication corresponds to concatenation of diagrams, so

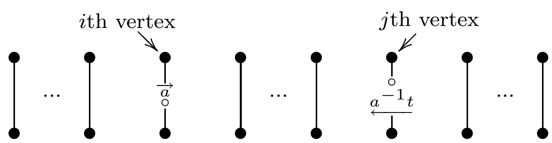


We generalize these diagrams to $GL_n(\mathbb{F}_q)$ by adding different varieties of "beads" to these diagrams that slide along the strands. A diagonal matrix corresponds to the identity permutation with a bead on each strand, such as



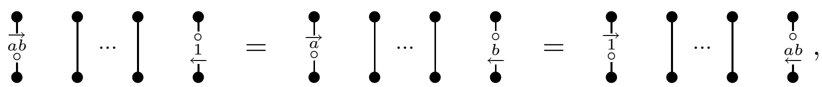

corresponds to $\quad \mathrm{diag}(h_1, h_2, \ldots, h_n)$,

and we depict the matrix $x_{ij}(t)$ by the identity permutation with directed beads on the $i$th and $j$th strands, such as



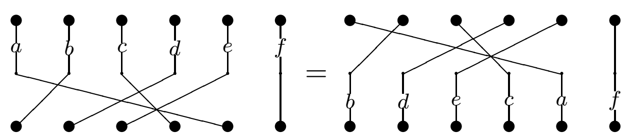

corresponds to $\quad x_{ij}(t) \quad$ for $a \in \mathbb{F}_q^*$.

Note there is an implicit relation in this last correspondence given by




for $a, b \in \mathbb{F}_q$.

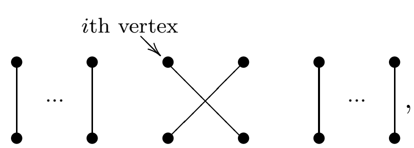
The advantage of this approach is that it allows a visual shortcut to computing products (such as the permutations above) and commutations in $GL_n(\mathbb{F}_q)$. For example, we can summarize multiple applications of (N3) by simply pushing the beads of $h \in T$ along the strands of $w \in W$ so that

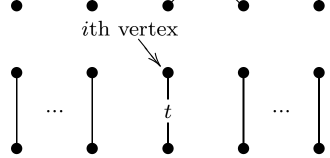


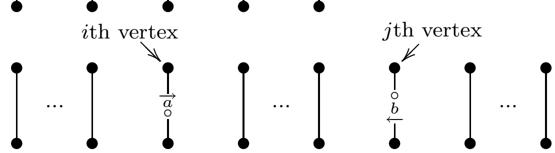

gives

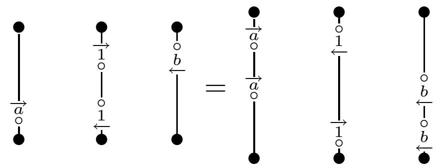
$$\begin{aligned}&s_4s_3s_4s_2s_3s_1h_{\varepsilon_1}(a)h_{\varepsilon_2}(b)h_{\varepsilon_3}(c)h_{\varepsilon_4}(d)h_{\varepsilon_5}(e)h_{\varepsilon_6}(f)\\ &\qquad\qquad = h_{\varepsilon_5}(a)h_{\varepsilon_1}(b)h_{\varepsilon_4}(c)h_{\varepsilon_2}(d)h_{\varepsilon_3}(e)h_{\varepsilon_6}(f)s_4s_3s_4s_2s_3s_1.\end{aligned}$$

The generators of $G$ are

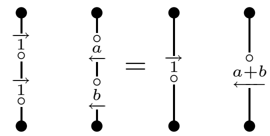
$s_i \quad$ as  , $\quad$ (5.4)


$h_{\varepsilon_i}(t) \quad$ as  , $\quad$ (5.5)


$x_{ij}(ab) \quad$ as  , $\quad$ (5.6) where each diagram has two rows of $n$ vertices. In the following Chevalley relations, curved strands indicate longer strands, so for example (UN1) indicates that $\underset{\circ}{\overset{\to}{a}}$ and $\overset{\circ}{\underset{\leftarrow}{b}}$ slide along the strands they are on (no matter how long). The Chevalley relations translate to

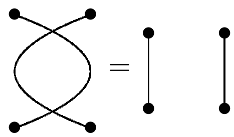


$(U1)$

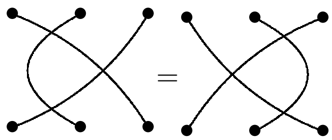


$(U2)$

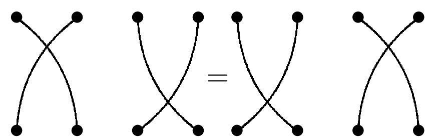
(the beads $\underset{\circ}{\overset{\to}{a}}$ and $\underset{\circ}{\overset{\to}{b}}$ commute unless two arrows or two circles encounter one-another on a strand).



$(N1)$

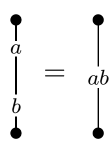


and

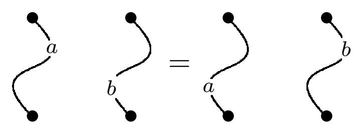


$(N2)$

(relations in $W$ exactly describe what one can do by pushing the strands around the diagrams),

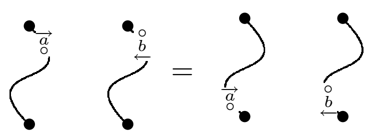


$(N4)$

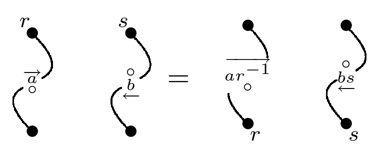


$(N5)$

($T$-type beads follow strands and multiply if they hit one-another),

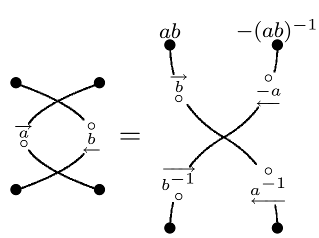


$(UN1)$



$(UN2)$

(beads $\underset{\circ}{\overset{\to}{a}}$ and $\overset{\circ}{\underset{\leftarrow}{b}}$ slide along strands unless they simultaneously hit a crossing (see (UN3) below), and the circle or arrow determine how $T$-type beads interact with $\underset{\circ}{\overset{\to}{a}}$ and $\overset{\circ}{\underset{\leftarrow}{b}}$),



$ab \neq 0. \qquad (UN3)$

(if $\underset{\circ}{\overset{\to}{a}}$ and $\overset{\circ}{\underset{\leftarrow}{b}}$ get stuck between two crossings, "things explode").

### 5.2 The unipotent Hecke algebra $\mathcal{H}_\mu$

Fix a nontrivial group homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$ , fix a map

$$\begin{array}{cccc} \mu: & \{1,2,\ldots,n-1\} & \longrightarrow & \{0,1\} \\ & i & \mapsto & \mu_i \end{array} \qquad \text{and define} \quad \mu_{ij} = \begin{cases} \mu_i, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.7}$$

Then

$$\begin{array}{rccc} \psi_\mu : & U & \longrightarrow & \mathbb{C}^* \\ & x_{ij}(t) & \mapsto & \psi(\mu_{ij}t) \end{array} \tag{5.8}$$

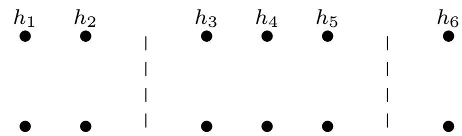
is a group homomorphism.

The unipotent Hecke algebra $\mathcal{H}_\mu$ of the triple $(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \mathrm{End}_G\left(\mathrm{Ind}_U^G(\psi_\mu)\right) \cong e_\mu \mathbb{C}G e_\mu, \quad \text{where} \quad e_\mu = \frac{1}{|U|}\sum_{u\in U} \psi_\mu(u^{-1})u. \tag{5.9}$$

If $N_\mu = \{v \in N \mid \ e_\mu v e_\mu \neq 0\}$, then $\{e_\mu v e_\mu \ \mid \ v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$.

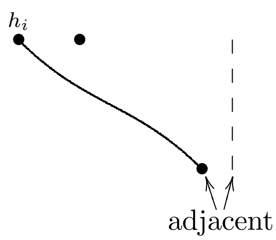
We may characterize the elements of $N_\mu$ in the following fashion (for a more extensive analysis of $N_\mu$ see [Th]). Suppose $v \in N$. For each $\mu_i = 0$, place a dotted line between the $i$th and $(i+1)$st vertices; for example, $\mu = (1, 0, 1, 1, 0, 0)$ gives



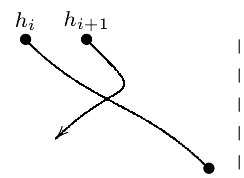

.

Then $e_\mu v e_\mu \neq 0$ if and only if the diagram for $v$ satisfies

(1) if

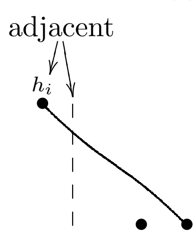



, then

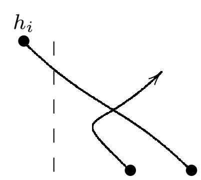



,

(2) if

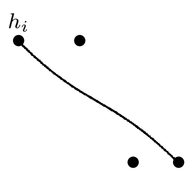



, then

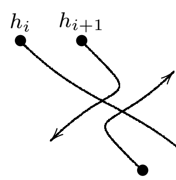



,

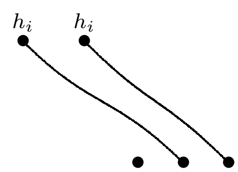
(3) if




, then either

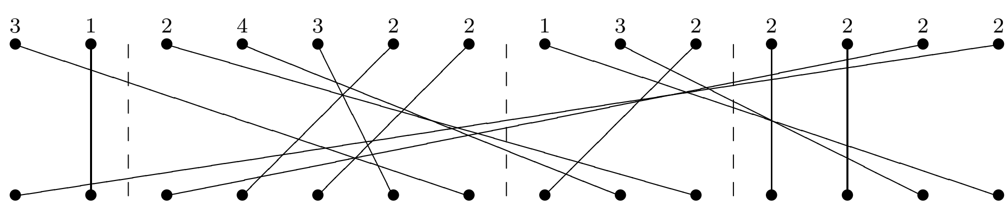



or




.

**Example.** If $\mu = (1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0)$ then




$\in N_\mu.$

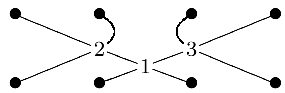
Note that the map

$$\begin{array}{rccl} \pi : N = WT & \longrightarrow & W & \\ wh & \mapsto & w, & \text{for } w \in W,\, h \in T, \end{array} \tag{5.10}$$

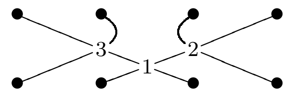
is a surjective group homomorphism. Let $u \in N$ with $\pi(u) = s_{i_1} \cdots s_{i_r}$ for $r$ minimal. Then there is a unique $u_T \in T$ such that

$$u = u_1 u_2 \dots u_r u_T, \qquad \text{where } u_k = s_{i_k}. \tag{5.11}$$

We write $u_k$ instead of $s_{i_k}$ because when working with diagrams, it is clear *where* the crossing is located and it is more important to determine the order in which order the crossings come, as in




or




.

For $t \in \mathbb{F}_q$, write $u_k(t) = s_{i_k} x_{i_k}(t)$.

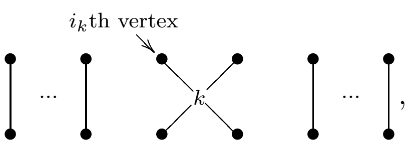
For any $\mu$ as in (5.7), the decomposition

$$U = \prod_{1 \le i < j \le n} U_{ij} \qquad \text{where} \quad U_{ij} = \langle x_{ij}(t) \mid t \in \mathbb{F}_q \rangle,$$

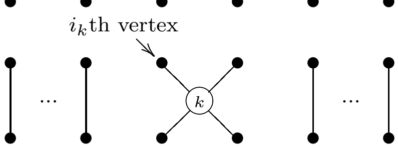
implies

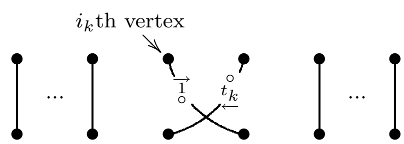
$$e_\mu = \prod_{1 \le i < j \le n} e_{ij}(\mu_{ij}), \qquad \text{where} \quad e_{ij}(k) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-kt) x_{ij}(t). \tag{5.12}$$

Pictorially,

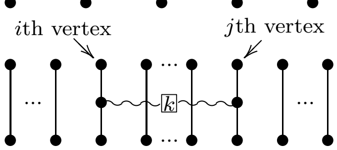
$u_k =$ 

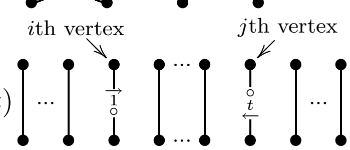
, (5.13)

$u_k(t_k) =$ 

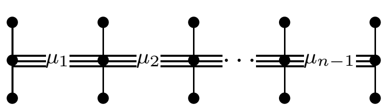
$=$ 

, (5.14)

$e_{ij}(k) =$ 

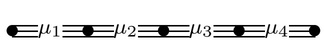
$= q^{-1} \sum_{t \in \mathbb{F}_q} \psi(-kt)$ 

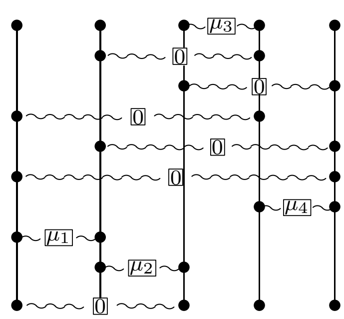
, (5.15)

$e_\mu =$ 

. (5.16)

Therefore, if n=5, since $e_\mu = e_{13}(0)e_{23}(\mu_2)e_{12}(\mu_1)e_{45}(\mu_4)e_{15}(0)e_{25}(0)e_{14}(0)e_{35}(0)e_{24}(0)e_{34}(\mu_3)$,



$=$ 

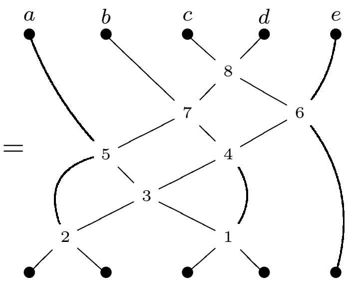
. (5.17)

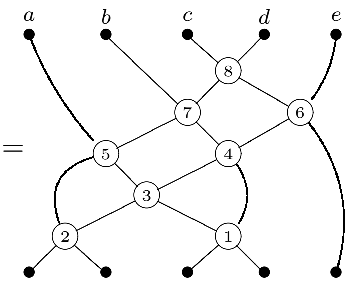
.

**A running example.** Throughout this section I will illustrate points using the example

$$u = u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_T \in N_5 \quad \text{according to} \quad s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in S_5, \tag{5.18}$$

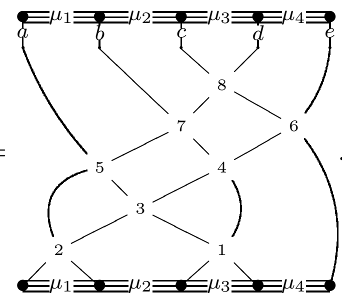
with $u_T = \mathrm{diag}(a, b, c, d, e)$. In this case,

$u =$ 

, $\quad u_1(t_1)u_2(t_2)\cdots u_8(t_8)u_T =$ 

,

and

$e_\mu u e_\mu =$ 

.

The elements $e_{ij}(k)$ also interact with $U$ and $N$ as follows (see also Section 4.1)

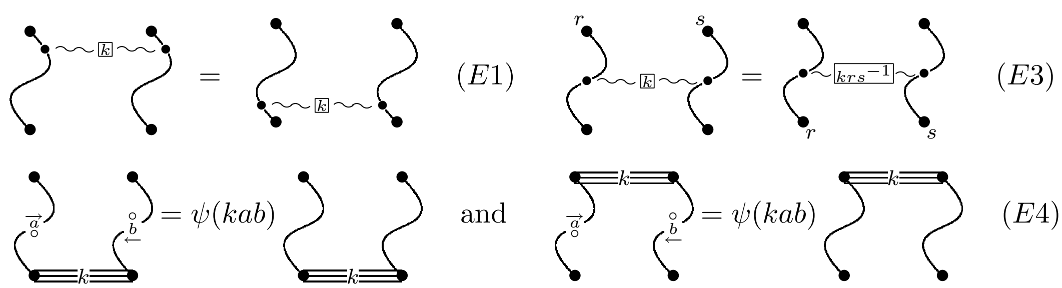
$$s_r e_{ij}(k) s_r = e_{s_r(i)s_r(j)}(k), \tag{E1}$$

$$e_\mu v e_{ij}(\mu_{ij}) = e_\mu v, \qquad v \in N_\mu, (\pi v)(i) < (\pi v)(j), \tag{E2}$$

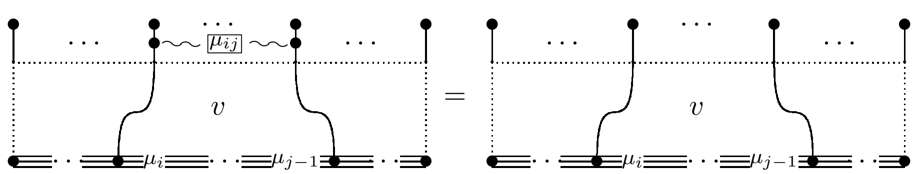
$$e_{ij}(k) h_{\varepsilon_l}(r) = h_{\varepsilon_l}(r) e_{ij}(k r^{\delta_{li}} r^{-\delta_{lj}}), \tag{E3}$$

$$e_\mu x_{ij}(t) = \psi(\mu_{ij} t) e_\mu = x_{ij}(t) e_\mu, \tag{E4}$$

or pictorially,



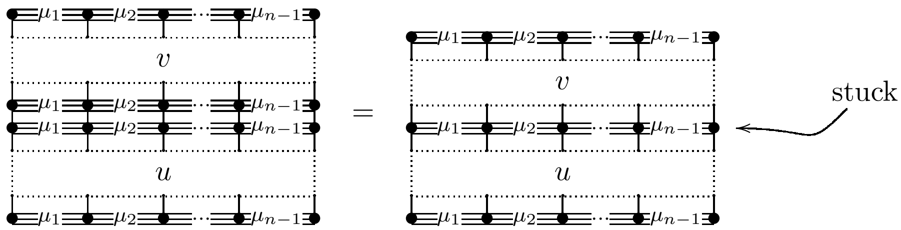

and for $v \in N_\mu$ with $(\pi v)(i) < (\pi v)(j)$,



$(E2)$

## 5.3 Basis element multiplication using braids

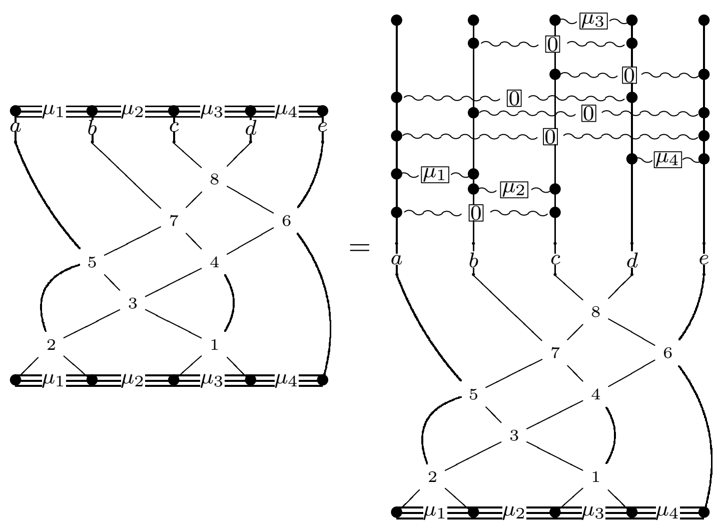
When we multiply two basis elements $e_\mu u e_\mu$ and $e_\mu v e_\mu$, the product $e_\mu u e_\mu v e_\mu$ has an $e_\mu$ "stuck" between the $u$ and the $v$, or




We then use the Chevalley relations to piece by piece "push" the center $e_\mu$ to the outside of the diagram. The first step is to push the $e_\mu$ as far into $u$ as possible, as illustrated by the following example.

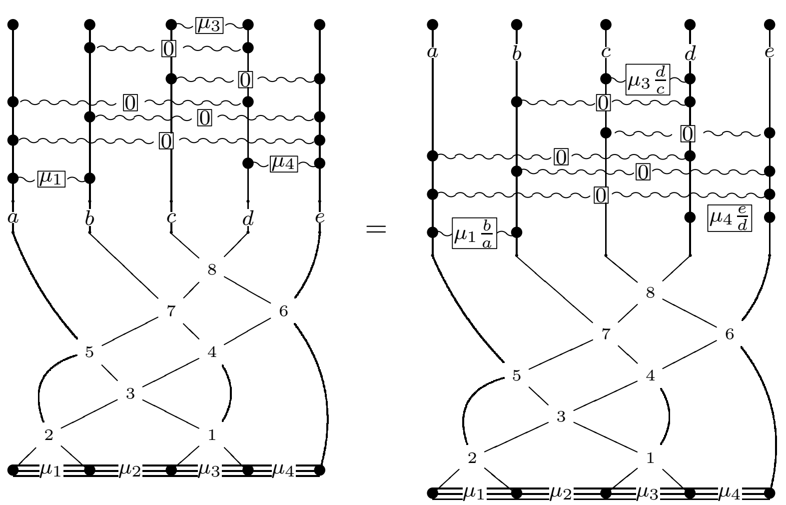
**Example (see (5.18)).** Let $u = u_1 u_2 \cdots u_8 u_T \in N$ according to $s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W$ and $u_T = \mathrm{diag}(a, b, c, d, e) \in T$. By (5.17), we may write

$e_\mu u e_\mu =$

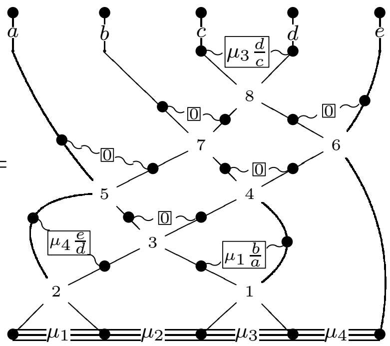
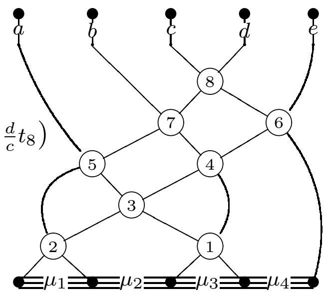


Note that the strands that $e_{13}(0)$ and $e_{23}(\mu_2)$ connect never cross, so we can use (E2) to push them through the diagram of $u$. The rest of the $e_{ij}(k)$ get stuck on some crossing, so we use (E3) to first move $u_T$ through the remaining $e_{ij}(k)$

$e_\mu u e_\mu =$



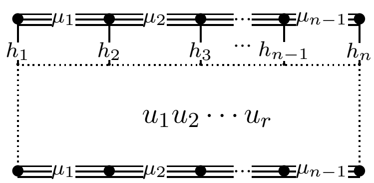
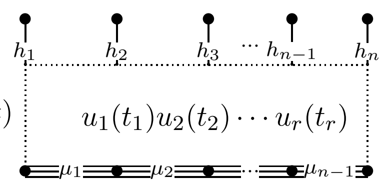
Next, use (E1) to push the $e_{ij}(k)$ down into $u$ until the strands they are on cross,

$=$  $= \dfrac{1}{q^8} \sum_{t\in\mathbb{F}_q^8} \psi\big(-\mu_1 \tfrac{b}{a} t_1 - \mu_4 \tfrac{e}{d} t_2 - \mu_3 \tfrac{d}{c} t_8\big)$  (5.19)

by definitions (5.15), (5.13), and (5.14).

**Step 0: Push $e_\mu$ into the diagram $u$**

. Suppose $u = u_1u_2\cdots u_r u_T \in N_\mu$ with $u_T = \mathrm{diag}(h_1, h_2, \ldots, h_n)$. As illustrated in the example above, use (5.12), (E3), (E1) and (E2) to rewrite $e_\mu u e_\mu$ as

 $= \dfrac{1}{q^r} \sum_{t\in\mathbb{F}_q^r} (\psi \circ f_u)(t)$  (5.20)

where $f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r]$ is given by

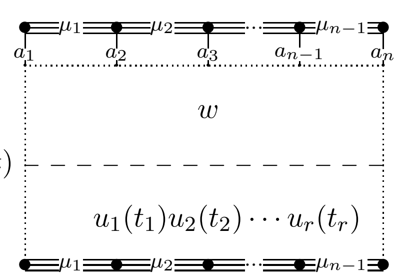
$$f_u(y_1, y_2, \ldots, y_r) = -\mu_{i_1j_1} h_{i_1}^{-1} h_{j_1} y_1 - \mu_{i_2j_2} h_{i_2}^{-1} h_{j_2} y_2 - \cdots - \mu_{i_rj_r} h_{i_r}^{-1} h_{j_r} y_r, \tag{5.21}$$

where $(i_k, j_k) = (a, b)$, if the $k$th crossing in $u$ crosses the strands coming from the $a$th and $b$th top vertices.

Note that relation (5.20) can be quickly computed by visually ascertaining which strands cross in the diagram.

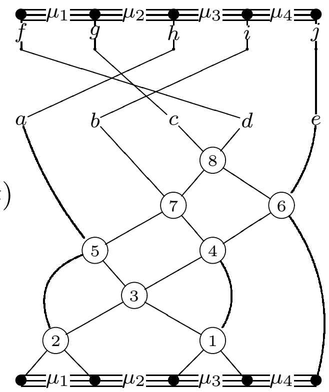
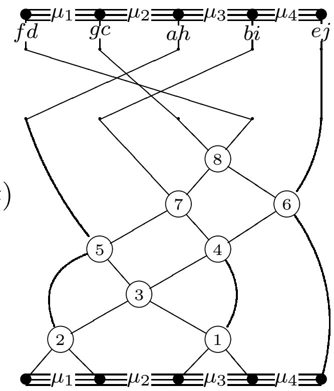
**Step 1: Concatenate $(e_\mu u e_\mu)$ with $(ve_\mu)$**

Let $u = u_1u_2\cdots u_r u_T \in N_\mu$ according to a minimal expression in $W$ as in (5.11). Let $v \in N_\mu$ and use (N3) and (N4) to write $u_Tv = w \cdot \mathrm{diag}(a_1, a_2, \cdots, a_n)$ where $w = \pi(v) \in W$ (see (5.7)). Then use (5.20) to write

$(e_\mu u e_\mu)(e_\mu v e_\mu) = (e_\mu u e_\mu)(v e_\mu) = \dfrac{1}{q^r} \sum_{t\in\mathbb{F}_q^r} (\psi \circ f_u)(t)$  (5.22)

(This form corresponds to $\Xi^{\emptyset}(u, u_Tv)$ of Corollary 3).

**Example (continued)**. If $u$ is as in (5.18) and $v = s_2s_3s_2s_1s_2 \cdot \mathrm{diag}(f,g,h,i,j) \in N$, then by (5.19) and (N3), $(e_\mu u e_\mu)(e_\mu v e_\mu) = (e_\mu u e_\mu)(v e_\mu)$ is equal to

$$\frac{1}{q^8}\sum_{t\in\mathbb{F}_q^8}(\psi\circ f_u)(t)$$  $$= \frac{1}{q^8}\sum_{t\in\mathbb{F}_q^8}(\psi\circ f_u)(t)$$  .

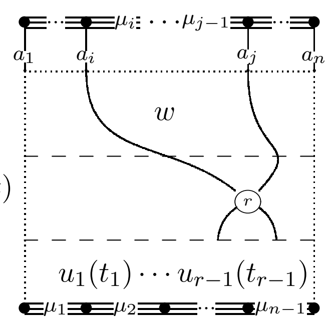
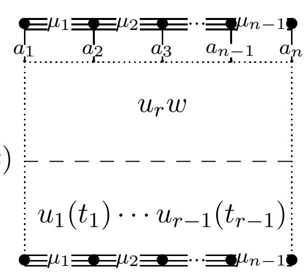
**Step 2: Apply "braid" relations**

Consider the crossing in (5.22) corresponding to $u_r(t_r)$ (the top crossing of $u$). There are two possibilities.

**Case 1** the strands that cross at $r$ do not cross again as they go up to the top of the diagram ($\ell(u_r w) > \ell(w)$),

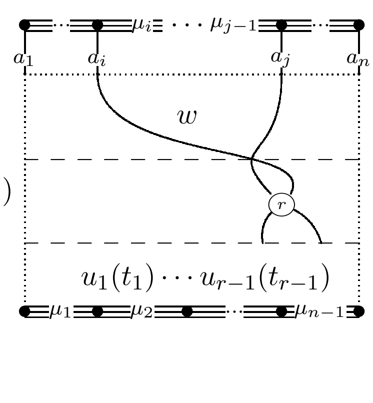
**Case 2** the strands that cross at $r$ cross once on the way up to the top of the diagram ($\ell(u_r w) < \ell(w)$).

**Relation 1 (Case 1):** by (UN1), (UN2) and (E4), $(e_\mu u e_\mu)(v e_\mu)$ is equal to

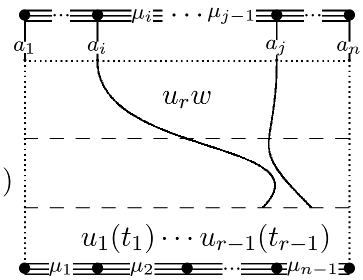
$$\frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f_u)(t)$$  $$= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f_u^{(+0)})(t)$$  (R1)

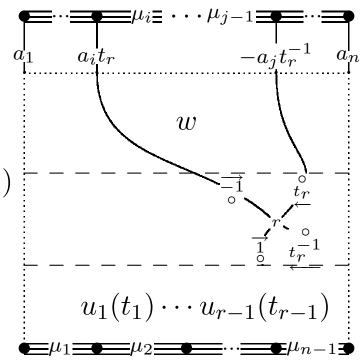
where $f_u^{(+0)} = f_u + \mu_{ij}a_i^{-1}a_jy_r$. Note that $f_u^{(+0)} = f_u$ unless $j = i+1$.

**Relation 2 (Case 2):** In case 2,

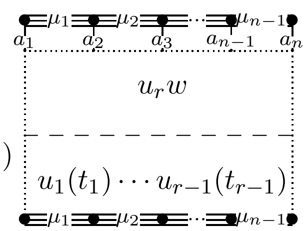
$$(e_\mu u e_\mu)(v e_\mu) = \frac{1}{q^r}\sum_{\substack{t'\in\mathbb{F}_q^{r-1}\\ t_r\in\mathbb{F}_q}}(\psi\circ f_u)(t',t_r)$$  .

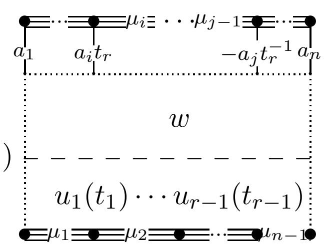
Split the sum into two parts corresponding to $t_r = 0$ and $t_r \neq 0$ to get

$$(e_\mu u e_\mu)(v e_\mu) = \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r = 0}} (\psi \circ f_u^{(-0)})(t', t_r)$$

(by (N1))

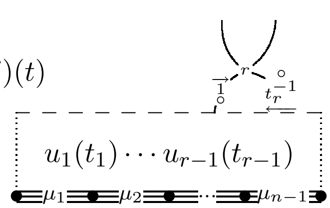
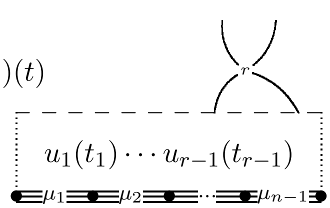
$$+ \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f_u)(t', t_r)$$

(by (UN3))

where $f_u^{(-0)} = f_u$. Use (UN1), (UN2), (U2), (U1) and (E4) on the second sum to push the pair $(\overset{\to}{\underset{\circ}{-1}}, \overset{\circ}{\underset{\leftarrow}{t_r}})$ to the top of the diagram and the pair $(\overset{\to}{\underset{\circ}{1}}, \overset{\circ}{\underset{\longleftarrow}{t_r^{-1}}})$ to the bottom.

$$(e_\mu u e_\mu)(v e_\mu) = \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r = 0}} (\psi \circ f_u^{(-0)})(t', t_r)$$

(R2)

$$+ \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f_u^{(1)})(t', t_r)$$


where $f_u^{(1)} = \varphi_r(f_u) + \mu_{ij} a_j a_i^{-1} y_r^{-1}$, and $\varphi_r(f)$ is defined by

$$\sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f)(t)$$

$$= \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ \varphi_r(f))(t)$$

. $(*)$

**Remarks:**

(a) We could have applied these steps for any $f$, $u$, and $v$, so we can iterate the process with each sum.

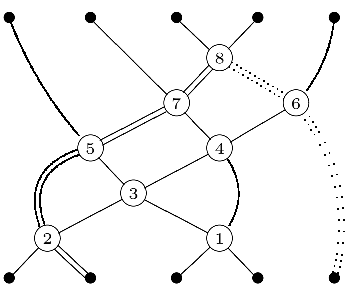
(b) The most complex step in these computations is determining $\varphi_r$. The following section develops a combinatorial method for computing the right-hand side of $(*)$.

**Step $2'$: A combinatorial way to compute $\varphi_k$.**

Relation $(*)$ pushes the beads $\underset{\circ}{\overset{\rightarrow}{1}}$ and $\overset{\circ}{\underset{\longleftarrow}{t_r^{-1}}}$ through the diagram until they get to the bottom. Along the way, the beads hit crossings and we either apply relation (U1), which leads to additional beads, or (U2), which forces us to renormalize. In the following, red paint corresponds to the strands traversed by beads of the form $\underset{\circ}{\overset{\rightarrow}{a}}$ and blue paint corresponds to strands traversed by beads of the form $\overset{\circ}{\underset{\leftarrow}{b}}$. Sinks encode places where we change $f$ (in $(*)$), while paths and their weights describe how to change $f$. Lemma 5.1 below gives the resulting evaluation of the map $\varphi_r$ in $(*)$ .

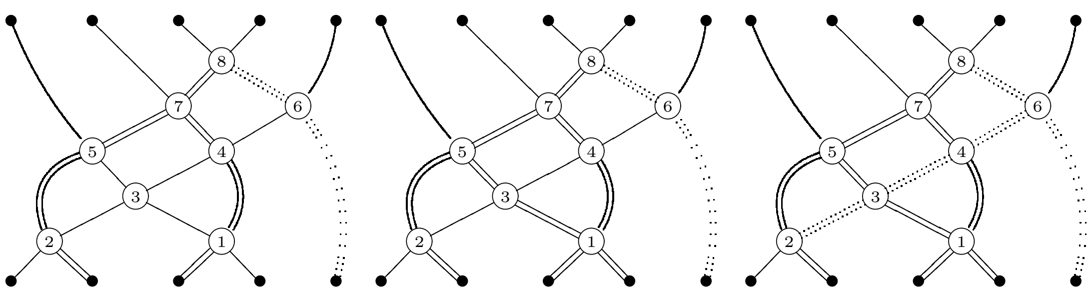
**Paint the strands below $u_k$ ($u^{\textcircled{k}}$).** Suppose $u = u_1 u_2 \cdots u_k \in N$ decomposes according to $s_{i_1} s_{i_2} \cdots s_{i_k} \in W$ (assume $u_T = 1$). Each step is illustrated with example (5.18).

(1) Paint the left [respectively right] strand exiting $\textcircled{k}$ below red [blue] all the way to the bottom of the diagram.



where red is $=$, blue is ⋯, and $\textcircled{k}$ is $\textcircled{8}$.

(2) For each crossing that the red [blue] strand passes through, paint the right [left] strand (if possible) red [blue] until that strand either reaches the bottom or crosses the blue [red] strand of (1).

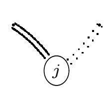


(5.23)

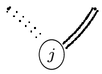
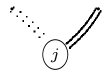
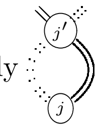
(3) Set

$$u^{\textcircled{k}} = \text{ the diagram } u_1(t_1)u_2(t_2)\cdots u_k(t_k) \text{ painted according to (1) and (2).} \tag{5.24}$$

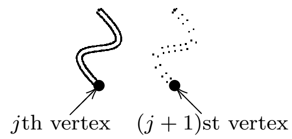
**Sinks**. The diagram $u^{\textcircled{k}}$ has a *crossed sink at* $\textcircled{j}$ if $\textcircled{j}$ is a crossing between a red strand and a blue one, or



Note that since $u$ is decomposed according to a minimal expression in $W$, there will be no crossings of the form

 (since  would imply  for some $j' \geq j$.)

The diagram $u^{\textcircled{k}}$ has a *bottom sink at* $j$ if a red strand enters $j$th bottom vertex *and* a blue strand enters the $(j+1)$st bottom vertex, or




**Example (continued)** In the running example above $u^{\textcircled{8}}$ has crossed sinks at $\textcircled{2}$, $\textcircled{3}$, and $\textcircled{4}$, and a bottom sink at 4. Note that $\textcircled{1}$ is not a crossed sink since both strands are red.

**Paths**. A *red [respectively blue] path* $p$ from a sink $s$ (either crossed or bottom) in $u^{\textcircled{k}}$ is an increasing sequence

$$j_1 < j_2 < \cdots < j_l = k,$$

such that in $u^{\textcircled{k}}$

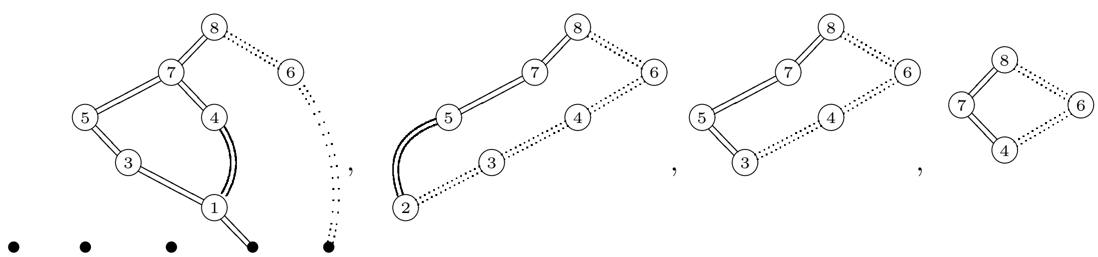
(a) $\textcircled{j_m}$ is directly connected (no intervening crossings) to $\textcircled{j_{m+1}}$ by a red [blue] strand,

(b) if $s$ is a crossed sink, then $\textcircled{j_1} = s$,

(b′) if $s$ is a bottom sink, then

- in a red path, the $s$th bottom vertex connects to the crossing $\textcircled{j_1}$ with a red strand.
- in a blue path, the $(s+1)$st bottom vertex connects to the crossing $\textcircled{j_1}$ with a blue strand.

**Example (continued).** The sinks with their corresponding paths for $u^{\textcircled{8}}$ are



Let

$$P_{=}(u^{\textcircled{k}}, s) = \left\{ \begin{matrix} \text{red paths from} \\ s \text{ in } u^{\textcircled{k}} \end{matrix} \right\} \quad \text{and} \quad P_{:}(u^{\textcircled{k}}, s) = \left\{ \begin{matrix} \text{blue paths from} \\ s \text{ in } u^{\textcircled{k}} \end{matrix} \right\} \tag{5.25}$$

The *weight* of a path $p$ is

$$\mathrm{wt}(p) = \begin{cases} \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} y_i, & \text{if } p \in P_{=}(u^{\textcircled{k}}, s), \\ \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} (-y_i), & \text{if } p \in P_{:}(u^{\textcircled{k}}, s). \end{cases} \tag{5.26}$$

Each sink $s$ in $u^{(k)}$ (either crossed $(j)$ or bottom $j$) has an associated polynomial $g_s \in \mathbb{F}_q[y_1, y_2, \ldots, y_{k-1}, y_k^{-1}]$ given by

$$g_s = \sum_{\substack{p \in P_=(u^{(k)}, s) \\ p' \in P_:(u^{(k)}, s)}} \mathrm{wt}(p) y_k^{-1} \mathrm{wt}(p'). \tag{5.27}$$

**Example (continued)**. Consider the weights of the above paths,

| Sink | 4 | 4 | 4 | ② | ② |
|---|---|---|---|---|---|
| Path | $1 < 3 < 5 < 7 < 8$ | $1 < 4 < 7 < 8$ | $6 < 8$ | $2 < 5 < 7 < 8$ | $2 < 3 < 4 < 6 < 8$ |
| Weight | $y_5$ | $y_1 y_7$ | $1$ | $1$ | $-y_6$ |

| Sink | ③ | ③ | ④ | ④ |
|---|---|---|---|---|
| Path | $3 < 5 < 7 < 8$ | $3 < 4 < 6 < 8$ | $4 < 7 < 8$ | $4 < 6 < 8$ |
| Weight | $y_5$ | $-y_6$ | $y_7$ | $-y_6$ |

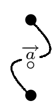
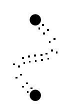
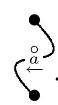
The corresponding polynomials are

$$g_4 = y_5 y_8^{-1} + y_1 y_7 y_8^{-1}, \quad g_{②} = -y_8^{-1} y_6, \quad g_{③} = -y_5 y_8^{-1} y_6, \quad g_{④} = -y_7 y_8^{-1} y_6. \tag{5.28}$$

**Lemma 5.1.** *Let $u = u_1 u_2 \cdots u_r$ and $\varphi_r$ be as in (R2) and $(*)$; suppose $u^{(r)}$ is painted as above. Then*

$$\varphi_r(f) = f \Big|_{\{y_j \mapsto y_j - g_{(j)} \ \mid \ (j) \text{ a crossed sink}\}} + \sum_{j \text{ a bottom sink}} \mu_j g_j.$$

*Proof.* In the painting,

 is a strand traveled by  and  is a strand traveled by .

Substitutions due to crossed sinks correspond to the normalizations in relation (U2), and the sum over bottom sinks comes from applications of relation (E4). ☐

For example (see (5.28)),

$$\varphi_8(f) = f\Big|_{\substack{y_4 \mapsto y_4 - g_{④} \\ y_3 \mapsto y_3 - g_{③} \\ y_2 \mapsto y_2 - g_{②}}} + \mu_4 g_4 = f\Big|_{\substack{y_4 \mapsto y_4 + y_7 y_8^{-1} y_6 \\ y_3 \mapsto y_3 + y_5 y_8^{-1} y_6 \\ y_2 \mapsto y_2 + y_8^{-1} y_6}} + \mu_4 (y_5 y_8^{-1} + y_1 y_7 y_8^{-1}).$$

## 5.4 A multiplication algorithm

**Theorem 5.1 (The algorithm).** *Let $G = GL_n(\mathbb{F}_q)$ and $u, v \in N_\mu$. An algorithm for multiplying $e_\mu u e_\mu$ and $e_\mu v e_\mu$ is*

*(1) Decompose $u = u_1 u_2 \cdots u_r u_T$ according to some minimal expression in $W$ (as in (5.11)).*

*(2) Put $e_\mu u e_\mu v e_\mu$ into the form specified by (5.22), with $u_T v = w \cdot \mathrm{diag}(a_1, a_2, \ldots, a_n)$ ($w = \pi(v) \in W$).*

*(3) Complete the following*

*(a) If $\ell(u_r w) > \ell(w)$, then apply relation (R1).*

*(b) If $\ell(u_r w) < \ell(w)$, then apply relation (R2), using $(u_1 u_2 \cdots u_r)^{ⓡ}$ and Lemma 5.1 to compute $\varphi_r$.*

*(4) If $r > 1$, then reapply (3) to each sum with $r := r - 1$ and with*

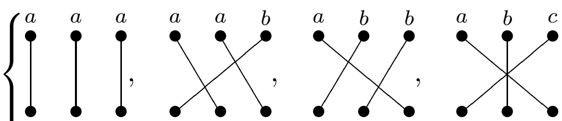
*(a) $w := u_r w$, after using (3a) or using (3b), in the first sum,*

*(b) $w := w$, after using (3b), in the second sum.*

*(5) Set all diagrams not in $N_\mu$ to zero.*

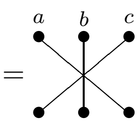
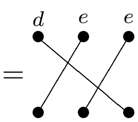
**Sample computation.** Suppose $n = 3$ and $\mu_i = 1$ for all $1 \leq i \leq 3$ (i.e.. the Gelfand-Graev case). Then

$$N_\mu = \Bigg\{$$



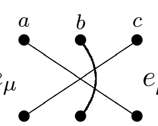
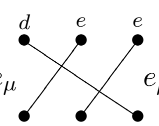
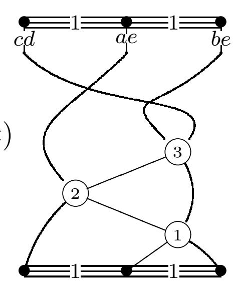
$$\Bigg| \quad a, b, c \in \mathbb{F}_q^* \Bigg\}$$

Suppose

$u =$  and $v =$  .

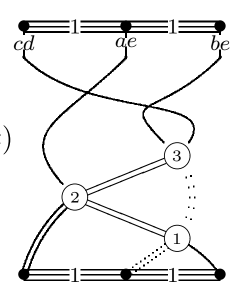
**1. Theorem 5.1 (1):** Let $u = u_1 u_2 u_3 u_T \in N_\mu$ decompose according to $s_2 s_1 s_2 \in W$, with $u_T = \mathrm{diag}(a, b, c)$.

**2. Theorem 5.1 (2):** By (5.22)

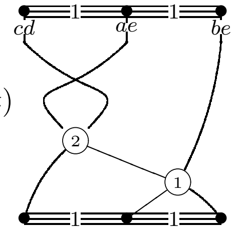
$(e_\mu$  $e_\mu)(e_\mu$  $e_\mu) = \dfrac{1}{q^3} \displaystyle\sum_{t \in \mathbb{F}_q^3} (\psi \circ f_u)(t)$ 

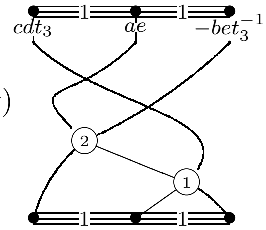
with $u_T v = s_2 s_1 \cdot \mathrm{diag}(cd, ae, be)$ (so $w = s_2 s_1$), and $f_u = -\frac{b}{a} y_1 - \frac{c}{b} y_3$ (as in (5.21)).

**3. Theorem 5.1 (3b):** Since $\ell(u_3 w) < \ell(w)$, paint $u_1(t_1) u_2(t_2) u_3(t_3)$ to get $(u_1 u_2 u_3)^{③}$ (as in (5.24)),

$= \dfrac{1}{q^3} \displaystyle\sum_{t \in \mathbb{F}_q^3} (\psi \circ f_u)(t)$ 

Now apply (R2),

$$= \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_3=0}} (\psi \circ f^{(-0)})(t)$$ 

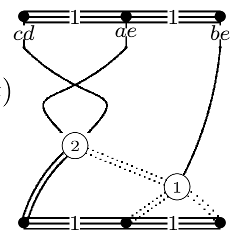
$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_3\in\mathbb{F}_q^*}} (\psi \circ f^{(1)})(t)$$ 

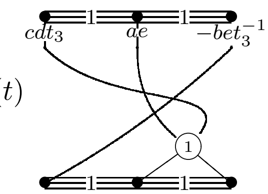

where $f^{(-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3$ and by Lemma 5.1,

$$f^{(1)} = \varphi_3(f_u) + \mu_{13}\frac{be}{cd}y_3^{-1} = -\frac{b}{a}y_1 + \frac{b}{a}y_2y_3^{-1} - \frac{c}{b}y_3 + y_3^{-1}.$$

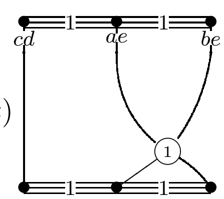
**4. Theorem 5.1 (4):** Set $r := 2$ with $w := u_r w = s_1$ in the first sum and $w := w$ in the second sum.

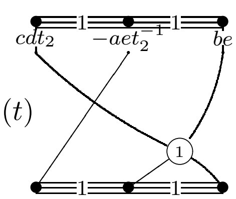
**5. Theorem 5.1 (3a) (3b):** In the first sum, $\ell(u_2s_1) < \ell(s_1)$, so paint $u_1(t_1)u_2(t_2)$ to get $(u_1u_2)^{②}$. In the second sum, $\ell(u_2s_2s_1) > \ell(s_2s_1)$, so apply (R1),

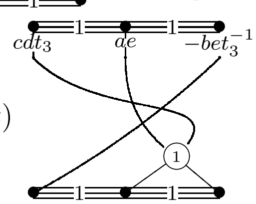
$$= \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_3=0}} (\psi \circ f^{(-0)})(t)$$ 

$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_3\in\mathbb{F}_q^*}} (\psi \circ f^{(+0,1)})(t)$$ 


where $f^{(+0,1)} = -\frac{b}{a}y_1 + \frac{b}{a}y_2y_3^{-1} - \frac{c}{b}y_3 + y_3^{-1} - \mu_3\frac{b}{a}y_2y_3^{-1} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + y_3^{-1}$. Now apply (R2) to the first sum,

$$= \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_2=t_3=0}} (\psi \circ f^{(-0,-0)})(t)$$ 

$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_2\in\mathbb{F}_q^*, t_3=0}} (\psi \circ f^{(1,-0)})(t)$$ 

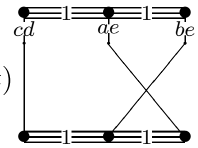

$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_3\in\mathbb{F}_q^*}} (\psi \circ f^{(+0,1)})(t)$$ 


where $f^{(-0,-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3$ and $f^{(1,-0)} = \varphi(f^{(-0)}) + \mu_1\frac{ae}{cd}y_2^{-1} = -\frac{b}{a}y_1 - y_2^{-1}y_1 + \frac{ae}{cd}y_2^{-1}$.

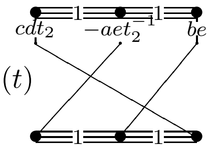
**6. Theorem 5.1 (4):** Set $r = 1$ with $w := u_2s_1 = 1$ in the first sum, $w := s_1$ in the second sum, and $w := s_1s_2s_1$ in the third sum.

**7. Theorem 5.1 (3a) (3a) (3b):** In the first sum $\ell(s_21) > \ell(1)$, so apply (R1); in the second sum $\ell(s_2s_1) > \ell(s_1)$, so apply (R1); in the third sum, $\ell(s_2s_1s_2s_1) < \ell(s_1s_2s_1)$, so paint $u_1(t_1)$ to get $u_1^{①}$,

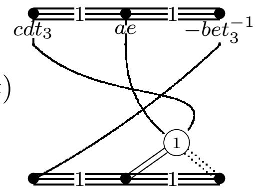
$$= \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2=t_3=0}} (\psi\circ f^{(+0,-0,-0)})(t)$$




$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2\in\mathbb{F}_q^*,t_3=0}} (\psi\circ f^{(+0,1,-0)})(t)$$

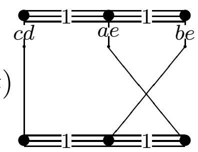



$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_3\in\mathbb{F}_q^*}} (\psi\circ f^{(+0,1)})(t)$$

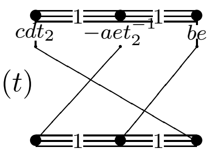



where $f^{(+0,-0,-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + \frac{b}{a}y_1 = -\frac{c}{b}y_3$ and $f^{(+0,1,-0)} = -\frac{b}{a}y_1 + \frac{ae}{cd}y_2^{-1} - y_2^{-1}y_1$. Now apply (R2) to the third sum

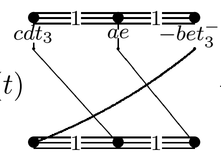
$$= \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2=t_3=0}} (\psi\circ f^{(+0,-0,-0)})(t)$$



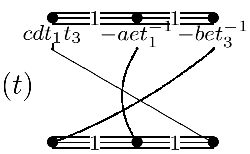

$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2\in\mathbb{F}_q^*,t_3=0}} (\psi\circ f^{(+0,1,-0)})(t)$$




$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1=0,t_3\in\mathbb{F}_q^*}} (\psi\circ f^{(-0,+0,1)})(t)$$




$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1,t_3\in\mathbb{F}_q^*}} (\psi\circ f^{(1,+0,1)})(t)$$

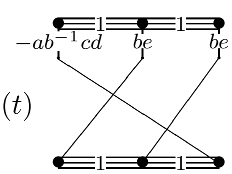



where $f^{(-0,+0,1)} = -\frac{c}{b}y_3 + y_3^{-1}$ and

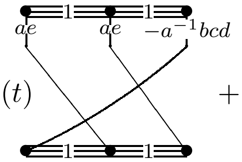
$$f^{(1,+0,1)} = \varphi_1(f^{(+0,1)}) + \mu_1\frac{ae}{cd}y_1^{-1}y_3^{-1} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + y_3^{-1} + y_1^{-1} + \frac{ae}{cd}y_1^{-1}y_3^{-1}.$$

**8. Theorem 5.1 (5):** The first sum contains no elements of $N_\mu$, so set it to zero. The second sum contains elements of $N_\mu$ when $be = -aet_2^{-1}$, so set $t_2 = -\frac{a}{b}$. The third sum contains elements of $N_\mu$ when $cdt_3 = ae$, so set $t_3 = \frac{ae}{cd}$. All the terms in the fourth sum are in $N_\mu$.

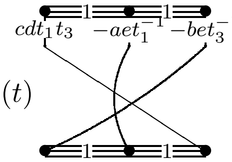
$$= 0 + \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2=-\frac{a}{b},t_3=0}} (\psi\circ f^{(+0,1,-0)})(t)$$



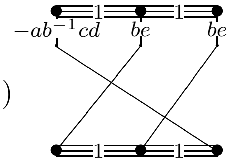

$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1=0,t_3=\frac{ae}{cd}}} (\psi\circ f^{(-0,+0,1)})(t)$$



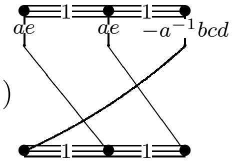

$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1,t_3\in\mathbb{F}_q^*}} (\psi\circ f^{(1,+0,1)})(t)$$



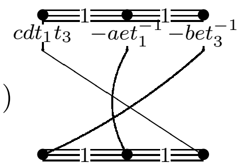

$$= \frac{1}{q^2}\psi(-\frac{be}{cd})$$




$$+ \frac{1}{q^2}\psi(-\frac{ae}{bd} + \frac{cd}{ae})$$



$$+\frac{1}{q^2}\sum_{t_1,t_3\in\mathbb{F}_q^*}\psi(-\frac{b}{a}t_1-\frac{c}{b}t_3+t_3^{-1}+t_1^{-1}+\frac{ae}{cd}t_1^{-1}t_3^{-1})$$

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