Math 156: Representation Theory

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Fall 2005

1 Introduction

Abstract algebraic structures such as groups, rings, and algebras have clear definitions but often lack simple, natural representations. To get around this problem we represent these abstract structures as concrete structures which we better understand, namely in matrices. This idea motivates the study of representation theory.

Example. *Consider the symmetric group* $S_n = \{\sigma : \{1,...,n\} \to \{1,...,n\} \mid \sigma \text{ is a bijection}\}$. *We can represent elements of* S_n *as matrices with exactly one 1 in every row and column and zeros elsewhere. For example,*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix} \qquad \mapsto \qquad \mathrm{perm}(\sigma) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

If $G = \{g_1, ..., g_n\}$ *is a finite group, then* G *can be viewed as a subgroup of* S_n, *where* $g_i \mapsto \sigma \in S_n$ *such that* $g_i g_j = g_{\sigma(j)}$. *It follows that we can view the elements of any finite group as* $n \times n$ *matrices.*

Some of the basic questions that this course will address include the following:

(1) What is a representation? What are some good ways to think of them?

(2) How can we classify representations? Is there a notion of "sameness" or "building blocks" of representations?

Applications of representation theory appear in mathematics (e.g. algebra, topology, geometry, probability, and combinatorics) as well as in physics, chemistry, and engineering.

2 Representations at a glance

Following the notation of the book[1] we write all functions to the *right* of their arguments, rather than to the left as usual. This will be somewhat confusing at first, but should get easier as we progress. Let $\mathbb{C}$ denote the complex numbers and define

$$M_n(\mathbb{C}) = \{n \times n \text{ matrices with entries in } \mathbb{C}\}.$$
$$GL_n(\mathbb{C}) = \{n \times n \text{ invertible matrices with entries in } \mathbb{C}\}.$$

1. James and Liebeck, *Representations and Characters of Groups*, 2nd edition

It is clear that $GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$. In algebraic terms, $M_n(\mathbb{C})$ is an additive abelian group and a multiplicative monoid, while $GL_n(\mathbb{C})$ is a multiplicative group.

Definition 1. *A* representation *is a map $\rho : G \to GL_n\mathbb{C}$ for some positive $n \in \mathbb{Z}$ such that $(g_1g_2)\rho = (g_1\rho)(g_2\rho)$ for all $g_1, g_2 \in G$.*

An important question to ask is why we use matrices to represent groups. A few key reasons include:

(1) We know quite a bit about linear algebra.

(2) Matrices provide a neutral setting in which to compare abstract groups.

(3) Linear transformations give a natural connection to vector spaces.

Example. *The following maps are both representations.*

(1) The map

$$\begin{array}{rccl} \rho : & G & \to & GL_n(\mathbb{C}) \\ & g & \mapsto & 1 \end{array}$$

is called the trivial representation*.*

(2) Consider the map

$$\begin{array}{rccccc} \rho : & S_n & \xrightarrow{\rho_1} & GL_n(\mathbb{C}) & \xrightarrow{\rho_2} & GL_1(\mathbb{C}) \\ & w & \mapsto & \mathrm{perm}(w) & \mapsto & \det(\mathrm{perm}(w)) \end{array}$$

where $\ker(\rho_1) = \{1\}$. Here $\ker(\rho_1\rho_2) = A_n$, the alternating group of permutations of $\{1, ..., n\}$, so $S_n/\ker(\rho_1\rho_2)$ is abelian.

3 Groups

A *group* G is a set with a map

$$\begin{array}{rccl} \cdot : & G \times G & \to & G \\ & (g,h) & \mapsto & gh \end{array}$$

such that

(1) $g(hk) = (gh)k$ for all $g, h, k \in G$.

(2) There exists an identity element $1 \in G$ such that $1 \cdot g = g \cdot 1 = g$ for all $g \in G$.

(3) For each $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.

A *subgroup* $H \subseteq G$ is a subset of G which is a group under the same map as G. A *group homomorphism* $\varphi : G \to H$ is a map such that

$$(gh)\varphi = (g\varphi)(h\varphi) \qquad \text{for all } g, h \in G.$$

The *kernel* of φ is the subgroup of G given by

$$\ker(\varphi) = \{g \in G \mid g\varphi = 1_H\} \subseteq G.$$

The *image* of φ is the subgroup of H given by

$$\mathrm{Im}(\varphi) = \{g\varphi \in H \mid g \in G\} \subseteq H.$$

An *isomorphism* is an invertible, i.e. bijective, group homomorphism.

4 Vector spaces

A *vector space* V over $\mathbb{C}$ is a set with maps

$$+: \begin{array}{ccc} V \times V & \to & V \\ (u,v) & \mapsto & u+v \end{array} \qquad \text{and} \qquad \cdot: \begin{array}{ccc} \mathbb{C} \times V & \to & V \\ (c,v) & \mapsto & cv \end{array}$$

such that for all $a, b \in \mathbb{C}$ and $u, v \in V$,

(1) $(a+b)v = av + bv$.

(2) $a(u+v) = au + av$.

(3) $a(bv) = (ab)v$.

(4) $1 \cdot v = v$.

(5) $u + v = v + u$.

A *subspace* $U \subseteq V$ is a subset of V that is a vector space under the same maps as V. A *basis* $\mathcal{B} = \{v_1, ..., v_n\}$ is a subset of V such that

(1) $V = \mathbb{C}\text{-span}\{v_1, ..., v_n\} = \{c_1 v_1 + ... c_n v_n \mid c_i \in \mathbb{C}\}$.

(2) $c_1 v_1 + ... c_n v_n = 0$ if and only if $c_1 = ... = c_n = 0$.

The *dimension* of V is $\dim(V) = |\mathcal{B}|$ where $\mathcal{B}$ is a basis for V.

Example. *A canonical example of a vector space is Euclidean space:* $\mathbb{C}^n = \mathbb{C}\text{-span}\{e_1, ..., e_n\}$.

5 Linear transformations

A *linear transformation* $T : U \to V$ is a map between vector spaces such that

$$(au + bv)T = a(uT) + b(vT) \qquad \text{for all } a, b \in \mathbb{C} \text{ and } u, v \in U.$$

Define

$$\begin{aligned} \mathrm{End}(V) &= \{T : V \to V \mid T \text{ is a linear transformation}\} \\ GL(V) &= \{T : V \to V \mid T \text{ is an invertible linear transformation}\} \end{aligned}$$

It is clear that $GL(V) \subseteq \mathrm{End}(V)$.

Remark. *If* $\mathcal{B} = \{v_1, ..., v_2\}$ *is a basis of a vector space* V*, then*

$$\theta: \begin{array}{ccc} GL(V) & \to & GL_n(\mathbb{C}) \\ T & \mapsto & \begin{pmatrix} v_1 T_{\mathcal{B}} \\ \vdots \\ v_n T_{\mathcal{B}} \end{pmatrix} \end{array}$$

is a group homomorphism. Here $(v_i T)_{\mathcal{B}} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}$ *if* $v_i T = c_1 v_1 + ... c_n v_n$.

Example. *If* $V = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$ *and*

$$\begin{aligned} v_1 T &= 5v_1 - 2v_2 \\ v_2 T &= v_3 \\ v_3 T &= 3v_1 + 3v_2 + 3v_3 \end{aligned}$$

then

$$T\theta = \begin{pmatrix} 5 & -2 & 0 \\ 0 & 0 & 1 \\ 3 & 3 & 3 \end{pmatrix}.$$

From this example is it apparent that $GL(V) \cong GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$ where $n = \dim(V)$.

6 Representations

Let G e a group. A *representation* ρ of G is a group homomorphism $\rho : G \to GL_n(\mathbb{C})$ for some positive $n \in \mathbb{Z}$. Equivalently, a representation ρ of G is a group homomorphisms $\rho : G \to GL(V)$. A *G-module* V is a vector space with a map $\circ : V \times G \to V$ such that for all $a, b \in \mathbb{C}$, $u, v \in V$, and $g, h \in G$,

(1) $(au + bv) \circ g = a(u \circ g) + b(v \circ g)$.

(2) $(V \circ g) \circ h = v \circ (gh)$.

(3) $v \circ 1 = v$.

Because of the importance these concepts, we repeat the preceding definition more formally.

Definition 2. *Let G be a group.*

(1) A representation *ρ of G is a group homomorphism $\rho : G \to GL_n(\mathbb{C})$ for some $n \in \mathbb{Z}_{>0}$.*

(2) A representation *ρ of G is a group homomorphism $\rho : G \to GL(V)$ for a vector space V.*

*(3) A G-*module *V is a vector space with a map $\circ : V \times G \to V$ such that for all $a, b \in \mathbb{C}$, $u, v \in V$, and $g, h \in G$,*

(a) $(au + bv) \circ g = a(u \circ g) + b(v \circ g)$.

(b) $v \circ 1 = v$.

(c) $(v \circ g) \circ h = v \circ (gh)$.

Remark. *Note that if $(V, \circ)$ is a G-module, then $0 \circ g = 0(0 \circ g) = 0$ for all $g \in G$.*

As the following proposition makes precise, the concepts of group representations and G-modules are essentially the same. Whenever we have a representation we have a natural G-module, and whenever we have a G-module we have a natural representation.

Proposition 6.1. *(2) and (3) are equivalent in the preceding definition. That is, there is a bijection between the set of representations of G and the set of G-modules.*

Proof. Suppose $\rho : G \to GL(V)$ is a representation, Then the map

$$\begin{array}{ccc} V \times G & \to & V \\ (v,g) & \mapsto & v(g\rho) \end{array} \qquad \text{for } v \in V, g \in G$$

makes V into a G-module. Conversely, suppose V is a G-module. Then the map

$$\begin{array}{cccc} \rho : & G & \to & GL(V) \\ & g & \mapsto & T_g \end{array}$$

is a representation, where T_g is the linear transformation given by

$$\begin{array}{cccc} T_g : & V & \to & V \\ & v & \mapsto & v \circ g \end{array}.$$

□

An advantage of G-modules over representations, and the primary reason why we consider them at all, is the fact that we do not need to specify an explicit basis in their construction. This relieves many of the notational difficulties involved with working with representations directly, while allowing us to prove the same results.

Definition 3. *A representation ρ is* faithful *if* $\ker(\rho) = \{1\}$. *A G-module is* faithful *if the corresponding representation is faithful.*

7 Some examples

When we write

$$G = \langle\ \overbrace{x_1, ..., x_n}^{\text{generators}} \mid \overbrace{R_1, ..., R_m}^{\text{relations}}\ \rangle$$

we mean

$$G = \{\text{Words in the letters } x_1, ..., x_n \text{ subject to the relations } R_1, ..., R_m\}.$$

The following groups will be helpful in a number of examples.

Definition 4. *The* cyclic group *of n elements is given by*

$$C_n = \langle\ x \mid x^n = 1\ \rangle.$$

Note that

$$\begin{array}{ccc} C_n & \xrightarrow{\cong} & \mathbb{Z}/n\mathbb{Z} \\ x^i & \mapsto & i \pmod n \end{array}.$$

Definition 5. *The* dihedral group *of $2n$ elements is given by*

$$D_{2n} = \langle\ s_1, s_2 \mid s_1^2 s_2^2 = 1,\ \underbrace{s_1 s_2 s_1 \ldots}_{n \text{ terms}} = \underbrace{s_2 s_1 s_2 \ldots}_{n \text{ terms}}\ \rangle.$$

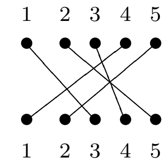
D_{2n} is the set of symmetries of a regular n-gon.

Definition 6. *We define the symmetric group S_n, the group of permutations of $\{1, ..., n\}$, by*

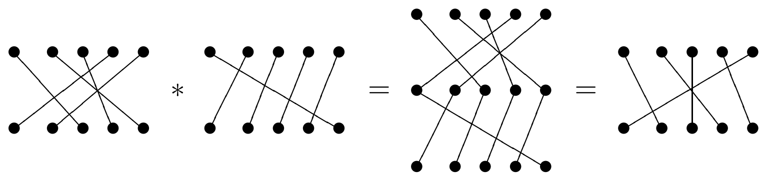
$$S_n = \langle\ s_1, ..., s_{n-1} \mid s_1^2 = ... = s_{n-1}^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ for } |i-j| > 1\ \rangle.$$

Example. *For $n = 5$, we can write a permutation*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix} \quad \mapsto$$

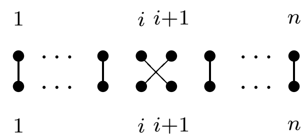


Writing permutations in this way, we multiply by concatenation:

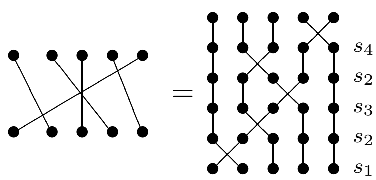


For $i = 1, ..., n-1$, let

$s_i =$



Note that



With this notation, it follows directly that

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \text{ if } |i-j| > 1$$

As one might already suspect, the symmetric group provides a wealth of examples in representation theory.

Example. *A representation of S_4 is given by*

$$\begin{array}{rccc} \rho: & S_4 & \to & GL_2(\mathbb{C}) \\ & s_1 & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ & s_2 & \mapsto & \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \\ & s_3 & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

The corresponding S_4-module is the vector space $V = \mathbb{C}\text{-span}\{v_1, v_2\}$ with the map $\circ : V \times S_4 \to V$ given by

$$\begin{array}{l} v_1 \circ s_1 = -v_1 \\ v_2 \circ s_1 = v_2 \end{array}, \qquad \begin{array}{l} v_1 \circ s_2 = -\frac{1}{2}v_1 + \frac{1}{2}v_2 \\ v_2 \circ s_2 = \frac{3}{2}v_1 + \frac{1}{2}v_2 \end{array}, \qquad \begin{array}{l} v_1 \circ s_3 = -v_1 \\ v_2 \circ s_3 = v_2 \end{array}$$

8 Isomorphic G-modules

Consider the following three S_4-modules:

$$\begin{array}{lll}
V_1 = \mathbb{C}\text{-span}\{v_1, v_2\} & V_2 = \mathbb{C}\text{-span}\{e_1, e_2\} & V_3 = \mathbb{C}^2 = \mathbb{C}\text{-span}\{u_1 = e_1 + e_2, u_2 = e_2 - e_1\} \\
s_1 : \begin{array}{l} v_1 \mapsto -v_1 \\ v_2 \mapsto v_2 \end{array} & s_1 : \begin{array}{l} e_1 \mapsto -e_1 \\ e_2 \mapsto e_2 \end{array} & s_1 : \begin{array}{l} u_1 \mapsto u_2 \\ u_2 \mapsto u_1 \end{array} \\
s_2 : \begin{array}{l} v_1 \mapsto -\frac{1}{2}v_1 + \frac{1}{2}v_2 \\ v_2 \mapsto \frac{3}{2}v_1 + \frac{1}{2}v_2 \end{array} & s_2 : \begin{array}{l} e_1 - \frac{1}{2}e_1 + \frac{1}{2}e_2 \\ e_2 \mapsto \frac{3}{2}e_1 + \frac{1}{2}e_2 \end{array} & s_2 : \begin{array}{l} u_1 \mapsto u_1 \\ u_2 \mapsto u_1 - u_2 \end{array} \\
s_3 : \begin{array}{l} v_1 \mapsto -v_1 \\ v_2 \mapsto v_2 \end{array} & s_3 : \begin{array}{l} e_1 \mapsto -e_1 \\ e_2 \mapsto e_2 \end{array} & s_3 : \begin{array}{l} u_1 \mapsto u_2 \\ u_2 \mapsto u_1 \end{array}
\end{array}$$

These are all *isomorphic* S_4-modules since there exist bijective linear transformations

$$\begin{array}{ccccc}
V_1 & \xrightarrow{\varphi_1} & V_2 & \xrightarrow{\varphi_2} & V_3 \\
v_1 & \mapsto & e_1 & \mapsto & \frac{1}{2}(u_1 - u_2) \\
v_2 & \mapsto & e_2 & \mapsto & \frac{1}{2}(u_1 + u_2)
\end{array}$$

which somehow preserves the operations of each module. This idea is made precise by the following definition.

Definition 7. *A G-modules $(U, \star)$ is* isomorphic *to a G-module $(V, \circ)$ if there exists a bijective linear transformation $\varphi : U \to V$ such that for all $u \in U$ and $g \in G$,*

$$(u \star g)\varphi = (u\varphi) \circ g.$$

Remark. *Suppose U and V are isomorphic G-modules.*

(1) We write $U \cong V$.

(2) If $U \cong V$ then $\dim(U) = \dim(V)$ (since bases are in bijection).

(3) Let $\rho : G \to GL_n(\mathbb{C})$ be a representation and suppose $U = \mathbb{C}\text{-span}\{u_1, ..., u_n\}$ and $V = \mathbb{C}\text{-span}\{v_1, ..., v_n\}$ such that $u_i \star g = u_i(g\rho)$ and $v_i \circ g = v_i(g\rho)$ for all $g \in G$. Then $U \cong V$ (the map $v_i \mapsto u_i$ is an isomorphism).

9 Equivalent representations

Given the equivalence of representations and G-modules, it follows directly that representations are "the same" if their corresponding G-modules are isomorphic. If this occurs, we say that two representations are *equivalent*.

Definition 8. *Two representations $\rho, \tau : G \to GL_n(\mathbb{C})$ are* equivalent *if they have isomorphic G-modules.*

Example. *The following representations of S_4 are equivalent:*

$$\begin{array}{rccc}
\rho : & S_4 & \to & GL_2(\mathbb{C}) \\
& s_1 & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
& s_2 & \mapsto & \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \\
& s_3 & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
\qquad
\begin{array}{rccc}
\tau : & S_4 & \to & GL_2(\mathbb{C}) \\
& s_1 & \mapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
& s_2 & \mapsto & \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \\
& s_3 & \mapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{array}$$

In fact for all $w \in S_4$,

$$(w\tau) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} (w\rho) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} (w\rho) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

This particular example suggests the following theorem.

Theorem 9.1. *Two representations* $\rho, \tau : G \to GL_n(\mathbb{C})$ *are equivalent if and only if there exists* $A \in GL_n(\mathbb{C})$ *such that*

$$(g\tau) = A(g\rho)A^{-1} \qquad \text{for all } g \in G.$$

Proof. Suppose $(g\tau) = A(g\rho)A^{-1}$ for all $g \in G$. Let $V_\rho = \mathbb{C}\text{-span}\{v_1, ..., v_n\}$ be a G-module corresponding to ρ and let $A = (A_{ij})$ Then the map

$$\begin{array}{ccc} V_\rho & \to & V_\rho \\ v_i & \mapsto & \sum A_{ij} v_j = u_j \end{array}$$

is a vector space isomorphism with $V_\tau \cong \mathbb{C}\text{-span}\{u_1, ..., u_n\}$.

Conversely, suppose τ, ρ are equivalent. Since the corresponding G-modules V_ρ and V_τ are isomorphic, we can assume that they are the same module with different bases. Let A be the transition matrix from one basis to another and let $V_\rho = \mathbb{C}\text{-span}\{v_1, ..., v_n\}$ and $V_\tau = \mathbb{C}\text{-span}\{u_1, ..., u_n\}$. Then

$$u_i(g\tau) = \sum_j A_{ij} v_j (g\rho) = \sum_{j,k} A_{ij} (g\rho)_{jk} v_k = \sum_{j,k,l} A_{ij} (g\rho)_{jk} \left(A^{-1}\right)_{kl} u_l.$$

Thus $(g\tau)_{il} = \sum_{j,k} A_{ij} (g\rho)_{jk} \left(A^{-1}\right)_{kl}$ so $(g\tau) = A(g\rho)A^{-1}$. □

Combining these past few results, we obtain the following diagram relating isomorphic G-modules and equivalent representations, and how we can move between them:

$$\begin{array}{ccc} \text{Isomorphic } G\text{-modules} & \leftrightarrow & \text{Equivalent representations} \\ \circlearrowleft & & \circlearrowleft \\ \text{Change of basis} & & \text{Conjugation} \end{array}$$

10 Submodules

Now that we have developed the basic ideas of representation theory, our plan is to refine the concepts of representations and G-modules into a notion of "smallest" or "prime" representations and modules. We shall then investigate when and how we can "factor" representations into these smaller parts. This plan leads naturally to the idea of a *submodule*.

Definition 9. *A* submodule $U \subseteq V$ *is a subspace such that* $u \circ g \in U$ *for all* $u \in U$ *and* $g \in G$.

A submodule is *proper* if $U \neq V$, while a submodule is *nontrivial* is $U \neq \{0\}$.

Definition 10. *Let* $(U, \circ)$ *and* $(V, \star)$ *be* G*-modules. A* G-module homomorphism $\varphi : U \to V$ *is a linear transformation such that for all* $u \in U$ *and* $g \in G$,

$$(u \circ g)\varphi = (u\varphi) \star g.$$

Lemma 10.1. *Let* $\varphi : U \to V$ *be a* G*-module homomorphism. The* $\ker(\varphi)$ *is a submodule of* U *and* $\mathrm{Im}(\varphi)$ *is a submodule of* V.

Proof. From linear algebra, we know that $\ker(\varphi)$ and $\mathrm{Im}(\varphi)$ are subspaces. We want to show that for all $g \in G$,

(1) $u \circ g \in \ker(\varphi)$ for all $u \in \ker(\varphi)$.

(2) $v \star g \in \mathrm{Im}(\varphi)$ for all $v \in \mathrm{Im}(\varphi)$.

To prove (1), note that if $u \in \ker(\varphi)$ and $g \in G$, then $(u \circ g)\varphi = (u\varphi) \star g = 0 \star g = 0$ so $u \circ g \in \ker(\varphi)$. To prove (2), note that if $x\varphi = v \in \mathrm{Im}(\varphi)$, then $v \star g = (x\varphi) \star g = (x \circ g)\varphi \in \mathrm{Im}(\varphi)$ so $v \star g \in \mathrm{Im}(\varphi)$. $\square$

The name we give to the "building blocks" of G-modules is *irreducible*.

Definition 11. *A G-module is* irreducible *if it has no proper, nontrivial submodules; otherwise, a G-module is* reducible. *A representation $\rho : G \to GL_n(\mathbb{C})$ is* irreducible *if V_ρ is irreducible; otherwise, a representation is* reducible.

Example. *Consider the S_4-modules $V = \mathbb{C}\text{-span}\{v_1, v_2, v_3, v_4\}$ where $v_i w = v_{iw}$ for $w \in S_4$. Then $u = \mathbb{C}\text{-span}\{v_1 + v_2 + v_3 + v_4\}$ is a proper nontrivial submodule. This module is one dimensional, so it has no nontrivial proper subspaces and is therefore irreducible.*

"Irreducible" is sometimes called "simple". What does this intuitively mean? Let $U \subseteq V$ be a nontrivial proper submodule. Let $U = \mathbb{C}\text{-span}\{v_1, ..., v_k\}$ and extend this basis so that $V = \mathbb{C}\text{-span}\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$. Then

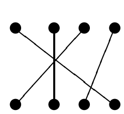
$$\begin{array}{rccc} \rho_V : & G & \to & GL_n(\mathbb{C}) \\ & g & \mapsto & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \end{array}$$

Is there a basis of V such that

$$\begin{array}{rccc} \rho_V : & G & \to & GL_n(\mathbb{C}) \\ & g & \mapsto & \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{array}$$

so that every $g\rho_V$ is conjugate to a block diagonal matrix? If there exists such a basis, then it is clear that the subspaces corresponding to the block diagonal parts of $g\rho_V$ are submodules, in which case we have a decomposition of V into a direct sum of submodules.

Example. *Let $V = \mathbb{C}\text{-span}\{v_1 + v_2 + v_3 + v_4, v_2, v_3, v_4\}$. Then $\rho_V : S_4 \to GL_4(\mathbb{C})$ maps*



$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Is there another basis which makes this matrix block diagonal?

11 Direct sums

Before pursuing these decomposition questions, we first review the idea of direct sums of vector spaces. Let $V_1, ..., V_r$ be vector spaces. Their *direct sum* is the vector space

$$V_1 \oplus ... \oplus V_r = \{(v_1, ..., v_r) \mid v_i \in V_i\}$$

where addition and scalar multiplication are defined by

(1) $c(v_1, ..., v_r) = (cv_1, ..., cv_r)$ for $c \in \mathbb{C}$.

(2) $(u_1, ..., u_r) + (v_1, ..., v_r) = (u_1 + v_1, ..., u_r + v_r)$ for $u_i, v_i \in V_i$.

Note that if $\mathcal{B}_i$ is a basis for V_i, the $V_1 \oplus ... \oplus V_r$ has a natural basis

$$\{(b_1, 0, ..., 0) \mid b_1 \in \mathcal{B}_1\} \cup \{(0, b_2, , 0..., 0) \mid b_2 \in \mathcal{B}_2\} \cup ... \cup \{(0, ...0, b_r) \mid b_r \in \mathcal{B}_r\}$$

and so $\dim(V_1 \oplus ... \oplus V_r) = \dim(V_1) + ... + \dim(V_r)$.

Example. *Let*

$$U = \mathbb{C}\text{-span}\{u_1, u_2\},$$
$$V = \mathbb{C}\text{-span}\{v_1\}.$$

Then $U \bigoplus V$ *has a basis* $\{(u_1, 0), (u_2, 0), (0, v_1)\}$ *and* $\dim(U \oplus V) = \dim(U) + \dim(V) = 3$.

Remark. *Let* $V_1, ..., V_r$ *be* G*-modules. Note that the map*

$$\begin{array}{rcl} V_1 \oplus ... \oplus V_r \times G & \to & V_1 \oplus ... \oplus V_r \\ ((v_1, ..., v_r), g) & \mapsto & (v_1 g, ..., v_r g) \end{array}$$

makes $V_1 \oplus ... \oplus V_r$ *into a* G*-module.*

We can think of the direct sum both as a way of constructing a new vector space from other vector spaces and a way of decomposing a vector space as a disjoint sum of its subspaces. In particular, if $V_1, ..., V_r \subseteq V$ are subspaces and $V_i \cap V_j = \{0\}$ for $1 \leq i < j \leq r$, then

$$\underbrace{\{v_1 + ... + v_r \mid v_i \in V_i\}}_{\text{inner direct sum}} \cong \underbrace{V_1 \oplus ... \oplus V_r}_{\text{outer direct sum}}.$$

We use the same $\oplus$ symbol to denote both types of direct sums.

Suppose $U \subseteq V$ is a nontrivial proper submodule. The main question we want to answer is whether there exists a submodule $W \subseteq V$ such that $V = U \oplus W$. As we shall soon see, the answer to this question is yes.

Let V be a vector space. A *projection* $\pi : V \to V$ is a linear transformation such that $\pi^2 = \pi$, i.e. for all $v \in V$, $(v\pi)\pi = v\pi$.

Proposition 11.1. *Let* $\pi : V \to V$ *be a projection. Then* $V = \operatorname{Im}(\pi) \oplus \ker(\pi)$.

Proof. We want to show that (1) if $v \in V$ then there exists $x \in \operatorname{Im}(\pi)$ and $y \in \ker(\pi)$ such that $v = x + y$, and (2) if $v \in \operatorname{Im}(\pi) \cap \ker(\pi)$ then $v = 0$. To prove (1), let $v \in V$. Note that $(v - v\pi)\pi = v\pi - v\pi^2 = 0$, so $v - v\pi \in \ker(\pi)$ and

$$v = \underbrace{v\pi}_{\in \operatorname{Im}(\pi)} + \underbrace{(v - v\pi)}_{\in \ker(\pi)}.$$

To prove (2), suppose $v \in \operatorname{Im}(\pi) \cap \ker(\pi)$. Then $v = v\pi = 0$, so $\operatorname{Im}(\pi) \cap \ker(\pi) = \{0\}$. $\square$

Corollary 11.1. *If* $\pi : V \to V$ *is a projection and* G*-module homomorphism, then* $\operatorname{Im}(\pi)$ *and* $\ker(\pi)$ *are submodules and* $V = \operatorname{Im}(\pi) \oplus \ker(\pi)$ *as* G*-modules.*

12 Maschke's theorem

Maschke's theorem answers in the affirmative the question of whether we can decompose a reducible G-module into submodules. It is this important result that makes the distinction between reducible and irreducible modules significant. Let G be a finite group.

Theorem 12.1. *(Maschke's Theorem) Let V be G-module. Suppose $U \subseteq V$ is a proper, non-trivial submodule. Then there exits a submodule $W \subseteq V$ such that $V = U \oplus W$.*

Remark. *This statement requires that we work in a field of characteristic zero such as $\mathbb{C}$ or $\mathbb{R}$.*

Proof. Let $U = \mathbb{C}\text{-span}\{v_1, ..., v_k\}$ and extend this basis to $V = \mathbb{C}\text{-span}\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$. Define

$$\begin{array}{rlll} \pi_U : & V & \to & V \\ & \sum_{i=1}^n a_i v_i & \mapsto & \sum_{i=1}^k a_i v_i \end{array}$$

Note that $\pi_U^2 = \pi_U$ so π_U is a projection, and $\mathrm{Im}(\pi_U) = U$. However, π_U is not necessarily a G-module homomorphism. To solve this problem, we use π_U to define a slightly different projection as follows. Define

$$\begin{array}{rlll} \pi : & V & \to & V \\ & v & \mapsto & \frac{1}{|G|}\sum_{g\in G} vg\pi_U g^{-1} \end{array}$$

If $u \in U$, then

$$u\pi = \frac{1}{|G|}\sum_{g\in G} \underbrace{(ug)}_{\in U} \pi_U g^{-1} = \frac{1}{|G|}\sum_{g\in G} ugg^{-1} = u.$$

Alternatively, if $v \in V$, then

$$\underbrace{(v\pi)}_{\in U}\pi = v\pi.$$

We conclude that π is a projection and that $\mathrm{Im}(\pi) = U$. Now let $h \in G$, and note that

$$(vh)\pi = \frac{1}{|G|}\sum_{g\in G} vhg\pi_U g^{-1} = \frac{1}{|G|}\sum_{x\in G} vx\pi_U x^{-1}h = (v\pi)h.$$

Thus π is a G-module homomorphism, so if $W = \ker(\pi)$ then $V = U \oplus W$ as G-modules. $\square$

The following corollary follows directly from the preceding result coupled with a bit of induction.

Corollary 12.1. *Let V be a G-module. Then there exist irreducible G-modules $V_1, ..., V_r$ such that $V = V_1 \oplus ... \oplus V_r$.*

Proposition 12.1. *Let V be a G-module and suppose*

$$\begin{array}{c} V = V_1 \oplus ... \oplus V_r \\ V = U_1 \oplus ... \oplus U_s \end{array}$$

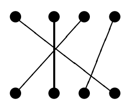
where $\{V_i\}$ and $\{U_j\}$ are irreducible G-modules. Then $r = s$ and there exists $w \in S_r$ such that $V_{iw} = U_i$ for all $1 \le i \le r$.

Proof. Note that $V_i \cap U_j$ is a submodule of V_i and U_j, so since V_i and U_j are irreducible, either $V_i \cap U_j = \{1\}$ or $V_i \cap U_j = V_i = U_j$. Since $V_i \subseteq V$, there must exist some j such that $V_i = U_j$, so define a map $w : \{1, ..., r\} \to \{1, .., r\}$ by $iw = j$ where $V_i = U_j$. Then w is injective since $U_j = U_{j'}$ implies $V_i = V_{i'}$ which in turn implies $i = i'$, so $w \in S_r$. $\square$

Example. *Consider the* S_4 *modules*

$$\begin{aligned} U &= \mathbb{C}\text{-span}\{v_1, v_2, v_3, v_4\} \\ V &= \mathbb{C}\text{-span}\{v_1 + v_2 + v_3 + v_4, v_2, v_3, v_4\} \end{aligned}$$

and the representation $\rho_V : S_4 \to GL_4(\mathbb{C})$ *that maps*

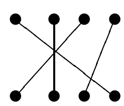


$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let

$$\begin{aligned} u_1 &= v_1 + v_2 + v_3 + v_4 \\ u_2 &= v_1 - v_2 \\ u_3 &= v_2 - v_3 \\ u_4 &= v_3 - v_4 \end{aligned}$$

Then $V = \mathbb{C}\text{-span}\{u_1, u_2, u_3, u_4\}$ *and* $\rho_V : S_4 \to GL_4(\mathbb{C})$ *maps*



$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In this case we can use Maschke's theorem to decompose V *as a direct sum of* $\mathbb{C}\text{-span}\{u_1\}$ *and* $\mathbb{C}\text{-span}\{u_2, u_3, u_4\}$.

Example. *For* p *prime, consider the representation*

$$\begin{array}{rccc} \rho : & C_p & \to & GL_2(\mathbb{Z}/p\mathbb{Z}) \\ & x^i & \mapsto & \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \end{array}$$

Then $V_\rho = \mathbb{Z}/p\mathbb{Z}\text{-span}\{v_1, v_2\}$ *has a submodule* $U = \mathbb{Z}/p\mathbb{Z}\text{-span}\{v_1\}$. *To see this let* $W = \mathbb{Z}/p\mathbb{Z}\text{-span}\{av_1 + bv_2\}$ *be a one dimensional submodule of* V_ρ. *Then we must have*

$$(av_1 + bv_2)x = (a + b)v_1 + bv_2 \in \mathbb{Z}/p\mathbb{Z}\text{-span}\{av_1 + bv_2\}$$

so $a + b = a$ *implies* $b = 0$ *and* $W = U$.

As we begin to develop more results, we should remember what our goals are. In particular, a model representation theorem for a finite group typically should do the following:

(1) Enumerate the irreducible G-modules / representations

(2) Determine their dimensions / degrees.

(3) Enumerate the corresponding irreducible characters.

With the tools we have developed so far, the first two of these goals should begin to make sense. In the next few sections, we shall begin to see how one goes about computing such things.

13 The group algebra

Let $\mathbb{C}G$ denote the vector space with basis G given by

$$\mathbb{C}G = \left\{ \sum_{g\in G} a_g g \mid a_g \in \mathbb{C} \right\}.$$

The action

$$\begin{array}{lcl} \mathbb{C}G \times G & \to & \mathbb{C}G \\ \left(\sum_{g\in G} a_g g, h\right) & \mapsto & \sum_{g\in G} a_g(gh) \end{array}$$

makes $\mathbb{C}G$ into a G-module. The corresponding representation $\rho_{\mathbb{C}G}$ is called the *regular representation.*

Example. *Consider $D_8 = \langle\ s_1, s_2 \mid s_1^2 s_2^2 = 1,\ s_1s_2s_1s_2 = s_2s_1s_2s_1\ \rangle$. Here we have*

$$\mathbb{C}D_8 = \{a_1 \cdot 1 + a_2s_1 + a_3s_2 + a_4s_1s_2 + a_5s_2s_1 + a_6s_1s_2s_1 + a_7s_2s_1s_2 + a_8s_1s_2s_1s_2\}.$$

G acts on $\mathbb{C}G$ by

$$(a \cdot 1 + bs_2s_1 + cs_1s_2s_1s_2)s_1 = as_1 + bs_2 + cs_2s_1s_2.$$

Proposition 13.1. *Let G be a nontrivial finite group. Then*

(1) $\dim(\mathbb{C}G) = |G|$.

(2) $\mathbb{C}G$ *is faithful.*

(3) $\mathbb{C}G$ *is reducible.*

Proof. (1) follows by definition. (2) follows from the fact that if $h \in \ker(\rho_{\mathbb{C}G})$ then $gh = g$ for all $g \in G$, so $h = 1$. To see (3), let $G = \{g_1, ..., g_n\}$ and define $\mathbb{1}_G = \mathbb{C}\text{-span}\{g_1 + ... + g_n\}$. Then $\mathbb{1}_G$ is a proper nontrivial submodule of $\mathbb{C}G$ (namely the *trivial* submodule of $\mathbb{C}G$) so $\mathbb{C}G$ is reducible. $\square$

Since $\mathbb{C}G$ is reducible, one naturally asks how the module reduces. The map

$$\begin{array}{lcl} \mathbb{C}G \times \mathbb{C}G & \to & \mathbb{C}G \\ \left(\sum_g a_g g, \sum_h b_h h\right) & \mapsto & \sum_{g,h} a_g b_h gh \end{array}$$

makes $\mathbb{C}G$ into a ring, and is this context we refer to $\mathbb{C}G$ as the *group algebra* of G over $\mathbb{C}$.

Definition 12. *A* group algebra *A is a vector space with a map*

$$\begin{array}{lcl} A \times A & \to & A \\ (x, y) & \mapsto & xy \end{array}$$

such that

$$(aw + bx)(cy + dz) = acwy + adwz + bcxy + bdxz$$

for all $a, b, c, d \in \mathbb{C}$ and $w, x, y, z \in A$.

Example. *The canonical example of a group algebra is $\mathbb{C}G$.*

Suppose V is a G module. Then the map

$$\begin{array}{lcl} V \times \mathbb{C}G & \to & V \\ \left(v, \sum_g a_g g\right) & \mapsto & \sum_g a_g vg \end{array}$$

makes V into a $\mathbb{C}G$-module. Conversely, if V is a $\mathbb{C}G$-module, then

$$\begin{array}{lcl} V \times G & \to & V \\ (v, g) & \mapsto & vg \end{array}$$

makes V into a G-module. In this way we can identify G-modules with $\mathbb{C}G$-modules, and this identification gives the bijections

$$\begin{array}{lcl} \rho : G \to GL(V) & \leftrightarrow & \rho : \mathbb{C}G \to \mathrm{End}(V) \\ \rho : G \to GL_n(\mathbb{C}) & \leftrightarrow & \rho : \mathbb{C}G \to M_n(\mathbb{C}) \end{array}$$

To decompose $\mathbb{C}G$ into irreducible modules, let $\mathbb{1}_G = \mathbb{C}\text{-span}\{e_{\mathbb{1}}\} \subseteq \mathbb{C}G$ where

$$e_{\mathbb{1}} = \frac{1}{|G|} \sum_{g \in G} g.$$

For $h \in G$,

$$e_{\mathbb{1}} h = \frac{1}{|G|} \sum_{g \in G} gh = \frac{1}{|G|} \sum_{g \in G} g = e_{\mathbb{1}}.$$

For this reason we call $\mathbb{1}_G$ the *trivial submodule* of $\mathbb{C}G$. As such, we have

$$\mathbb{C}G = \mathbb{1}_G \oplus ...$$

Note that $e_{\mathbb{1}} \in \mathbb{C}G \subseteq \mathrm{End}(\mathbb{C}G)$ is the linear transformation

$$\begin{array}{rlcl} e_{\mathbb{1}} : & \mathbb{C}G & \to & \mathbb{C}G \\ & \sum_{g \in G} a_g g & \mapsto & \left(\sum_{g \in G} a_g\right) e_{\mathbb{1}} \end{array}$$

so $\mathrm{Im}(e_{\mathbb{1}}) = \mathbb{1}_G$. Furthermore, if $v \in \mathbb{C}G$, then $vge_{\mathbb{1}} = ve_{\mathbb{1}} = ve_{\mathbb{1}}g$ so $e_{\mathbb{1}}$ is a G-module homomorphism. Lastly, $e_{\mathbb{1}}^2 = e_{\mathbb{1}} e_{\mathbb{1}} = \frac{1}{|G|} \sum_{g \in G} e_{\mathbb{1}} g = \frac{1}{|G|} \sum_{g \in G} e_{\mathbb{1}} = e_{\mathbb{1}}$, so

$$\mathbb{C}G = \mathrm{Im}(e_{\mathbb{1}}) \oplus \ker(e_{\mathbb{1}}) = \mathbb{1}_G \oplus \left\{ \sum_{g \in G} a_g g \mid \sum_{g \in G} a_g = 0 \right\}.$$

If $G = S_n$, then this expression becomes

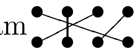
$$\mathbb{C}S_n = \mathbb{1}_{S_n} \oplus \left\{ \sum_{w \in S_n} a_w w \mid \sum_{w \in S_n} a_w = 0 \right\}.$$

Let $V^{(\mathbb{1}^n)} = \mathbb{C}\text{-span}\left\{e_{(\mathbb{1}^n)}\right\}$ where

$$e_{(\mathbb{1}^n)} = \frac{1}{n!} \sum_{w \in S_n} \mathrm{sgn}(w) w.$$

Here $\mathrm{sgn}(w)$ denotes the sign of the permutation $w \in S_n$, which can be defined as

$$\mathrm{sgn}(w) = (-1)^{l(w)}$$

where $l(w)$ denotes the number of crossings in the diagram . Note that for $g \in S_n$,

$$e_{(\mathbb{1}^n)} g = \frac{1}{n!} \sum_{w \in S_n} \mathrm{sgn}(w) w g = \frac{1}{n!} \sum_{w \in S_n} \mathrm{sgn}\left(w g^{-1}\right) w = \mathrm{sgn}\left(g^{-1}\right) e_{(\mathbb{1}^n)}$$

As a result, $e_{(\mathbb{1}^n)}$ is a projective $\mathbb{C}G$-module homomorphism with $\mathrm{Im}(e_{(\mathbb{1}^n)}) = V^{(\mathbb{1}^n)}$. It follows that

$$\mathbb{C}S_n = \mathbb{1}_{S_n} \oplus V^{(\mathbb{1}^n)} \oplus \left\{ \sum_{w \in S_n} a_w w \mid \sum_{w \in S_n} a_w = \sum_{w \in S_n} \mathrm{sgn}(w) a_w = 0 \right\}.$$

If $\left\{g_1, ..., g_{n!/2}\right\} \subseteq S_n$ is the set of even permutations and $\left\{h_1, ..., h_{n!/2}\right\} \subseteq S_n$ is the set of odd permutations, then

$$\mathbb{C}S_n = \mathbb{1}_{S_n} \oplus V^{(\mathbb{1}^n)} \oplus \mathbb{C}\text{-span}\left\{g_1 - g_2, ..., g_{n!/2-1} - g_{n!/2}, h_1 - h_2, ..., h_{n!/2-1} - h_{n!/2}\right\}.$$

For $n = 3$ this gives

$$\mathbb{C}S_n = \mathbb{1}_{S_n} \oplus V^{(\mathbb{1}^n)} \oplus \underbrace{\mathbb{C}\text{-span}\left\{g_1 - g_2, g_2 - g_3, h_1 - h_2, h_2 - h_3\right\}}_{\text{4-dimensional}}$$

and in fact there exists a 2-dimensional irreducible $\mathbb{C}S_3$-module.

14 Schur's lemma

Let V be a G-module and let $g \in G$. If there exists $v \in V$ and $\lambda \in \mathbb{C}$ such that $vg = \lambda v$, then v is called an *eigenvector* with *eigenvalue* λ of g. More specifically, the roots of $\det(g\rho_V - tI)$ are the eigenvalues of g.

Schur's lemma states that homomorphisms between irreducible G-modules are given either by scalar multiplication or the trivial mapping. Intuitively, this result means that irreducible G-modules are either essentially identical or totally incomparable. In this sense, Schur's lemma clarifies the idea of irreducible G-modules as atomic, interchangeable building blocks of representations.

Theorem 14.1. *(Schur's Lemma) Let U and V be irreducible G-modules Then*

(1) If $\varphi : U \to V$ is a G-module homomorphism, then either $\mathrm{Im}(\varphi) = \{0\}$ *or φ is an isomorphism.*

(2) If $\varphi : V \to V$ is a G-module isomorphism, then there exists $\lambda \in \mathbb{C}$ such that $v\varphi = \lambda v$ for all $v \in V$.

Proof. Let $\varphi : U \to V$ be a G-module homomorphism. (1) follows the fact that $\ker(\varphi)$ is a submodule of U, so is either U or $\{0\}$, and $\mathrm{Im}(\varphi)$ is a submodule of V, so is either V or $\{0\}$. This means that either φ is bijective or $\mathrm{Im}(\varphi) = \{0\}$.

Now let $\varphi : V \to V$ be a G-module isomorphism. To prove (2), let $n = \dim(V)$ and view $\varphi \in M_n(\mathbb{C})$. Since $\det(\varphi - tI)$ has a root $\lambda \in \mathbb{C}$, there exists $v \in V$ such that $v\varphi = \lambda v$. In this case $\ker(\varphi - \lambda I) \subseteq V$ is a nonzero submodule, so $\varphi = \lambda I$ and $v\varphi = \lambda v$ for all $v \in V$. $\square$

The following proposition gives a converse to Schur's lemma.

Proposition 14.1. *If for each G-module homomorphism $\varphi : V \to V$ there exists $\lambda \in \mathbb{C}$ such that $v\varphi = \lambda v$ for all $v \in V$, then V is irreducible.*

Proof. Suppose $U \subseteq V$ is a nontrivial proper submodule, so $V = U \oplus W$ for some submodule W. Let $\pi : U \oplus W \to U \oplus W$ be the projection given by $(u+w)\pi = u$ for $u \in U$ and $w \in W$. This map is a G-module homomorphism, so $u = (u+w)\pi = \lambda(u+w)$ for all $u \in U$ and $w \in W$. by Schur's lemma. Since $U \cap W = \{0\}$, this is only possible if $W = \{0\}$ and $\lambda = 1$, in which case U is not proper, or if $U = \{0\}$ and $\lambda = 0$, in which case U is trivial. From these contradictions we conclude that V is irreducible. □

Corollary 14.1. *A representation $\rho : G \to GL_n(\mathbb{C})$ is irreducible if and only if for every $A \in M_n(\mathbb{C})$ such that $(g\rho)A = A(g\rho)$ for all $g \in G$, there exists $\lambda \in \mathbb{C}$ such that $A = \lambda I$.*

Corollary 14.2. *G is abelian if and only if all of its representations are 1-dimensional.*

Proof. Suppose G is abelian and let V be an irreducible G-module. Let $x \in G$ and define

$$\begin{array}{rccc} R_x : & V & \to & V \\ & v & \mapsto & vx \end{array}$$

For $g \in G$,

$$(vR_x)g = vxg = vgx = (vg)R_x$$

so R_x is a nontrivial G-module homomorphism. By Schur's lemma, there exists $\lambda_x \in \mathbb{C}$ such that $vR_x = \lambda_x v$ for all $v \in V$, in which case $\mathbb{C}$-span$\{v\} \subseteq V$ is a nontrivial submodule so $V = \mathbb{C}$-span$\{v\}$ is 1-dimensional.

Conversely, suppose all G-modules are 1-dimensional. Then

$$\mathbb{C}G = V_1 \oplus ... \oplus V_{|G|}$$

where $V_i = \mathbb{C}$-span$\{v_i\}$ for some $v_i \in V$ for each i. It follows that $\{v_1, ..., v_{|G|}\}$ is a basis for $\mathbb{C}G$, and with respect to this basis the regular representation is given by

$$\begin{array}{rccc} \rho_{\mathbb{C}G} : & G & \to & GL_{|G|}(\mathbb{C}) \\ & g & \mapsto & \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{|G|} \end{pmatrix} \end{array}$$

Since $\mathbb{C}G$ is a faithful module and diagonal matrices commute, G must be abelian. □

15 The center

The *centers* of G and $\mathbb{C}G$ are defined respectively by

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

$$Z(\mathbb{C}G) = \{x \in \mathbb{C}G \mid xy = yx \text{ for all } y \in \mathbb{C}G\}$$

Remark. *We can make the following observations regarding $Z(G)$ and $Z(\mathbb{C}G)$:*

(1) $Z(G) \subseteq Z(\mathbb{C}G)$.

(2) $\mathbb{C}$-span$\{Z(G)\} \subseteq Z(\mathbb{C}G)$.

(3) $e_{\mathbb{1}} = \frac{1}{|G|}\sum_{g\in G} g \in Z(\mathbb{C}G)$.

(4) If $H \lhd G$ *then* $e_H = \frac{1}{|H|}\sum_{h\in H} h \in Z(\mathbb{C}G)$.

(5) Let $g \in G$. *Then* $\sum_{h\in G} hgh^{-1} \in Z(\mathbb{C}G)$

The center of $\mathbb{C}G$ is somehow "contained" in many other important structures within the decomposition of $\mathbb{C}G$ into irreducible modules. This property combined with the following propositions will allow us to prove several useful theorems when we start into character theory.

Proposition 15.1. *Suppose G has a faithful irreducible G-module. Then $Z(G)$ is cyclic.*

Proof. Let $z \in Z(G)$. Since

$$\begin{array}{rccc} R_z : & V & \to & V \\ & v & \mapsto & vz \end{array}$$

is a G-module homomorphism, there exists $\lambda_z \in \mathbb{C}$ such that $vz = \lambda_z v$ for all $v \in V$. Therefore the map

$$\begin{array}{rlcl} \varphi : & Z(G) & \to & GL_1(\mathbb{C}) \\ & z & \mapsto & \lambda_z \end{array}$$

is a faithful representation, so $Z(G)$ is isomorphic to a subgroup of $GL_1(\mathbb{C}) \cong \mathbb{C}^\times$, and so $Z(G)$ is given by $\cong \left\langle e^{\frac{2\pi i}{n}} \right\rangle$ for some n, which is cyclic. □

Proposition 15.2. *Let $\mathcal{K}_1, ..., \mathcal{K}_r$ be the distinct conjugacy classes of G, and define*

$$z_i = \sum_{g\in\mathcal{K}_i} g \in \mathbb{C}G.$$

Then $\{z_1, ..., z_r\}$ is a basis for $Z(\mathbb{C}G)$.

Proof. Note that for all $h \in G$,

$$h^{-1}z_i h = \sum_{g\in\mathcal{K}_i} h^{-1}gh = \sum_{k\in\mathcal{K}_i} k = z_i$$

so $z_i h = hz_i$. It follows that for all $x = \sum_{h\in G} a_h h \in \mathbb{C}G$, $z_i x = xz_i$, so each $z_i \in Z(\mathbb{C}G)$. To see that $z_1, ..., z_r$ are linearly independent, note that if $\sum_{i=1}^r \lambda_i z_i = 0$ for some $\lambda_i \in \mathbb{C}$, then we must have $\lambda_i = 0$ for all i since the conjugacy classes $\mathcal{K}_1, ..., \mathcal{K}_r$ are mutually disjoint. Finally, to show that $z_1, ..., z_r$ span $Z(\mathbb{C}G)$, suppose $r = \sum_{g\in G} a_g g \in Z(\mathbb{C}G)$. For $h \in G$,

$$h^{-1}rh = \sum_{g\in G} a_g h^{-1}rh = \sum_{g\in G} a_g g$$

so comparing coefficients it follows that $a_{h^{-1}gh} = a_g$ for all $g, h \in G$. Letting $g_1, ..., g_r$ be a set of representatives for $\mathcal{K}_1, ..., \mathcal{K}_r$, it follows that

$$r = \sum_{i=1}^r a_{g_i} z_i$$

so $\{z_1, ..., z_r\}$ is a basis for $Z(\mathbb{C}G)$. □

16 The G-module $\mathbb{C}G$

We repeat the proof of the following proposition using Schur's lemma.

Proposition 16.1. *Let V be a G-module and suppose $V = V_1 \oplus ... \oplus V_r$ is a decomposition into irreducible G-modules. If $U \subseteq V$ is an irreducible submodule, then $U \cong V_i$ for some i.*

Proof. For every $u \in U$,

$$u = v_1^{(u)} + ... + v_r^{(u)}$$

is a unique decomposition with $v_i^{(u)} \in V_i$ for $i \in \{1, ..., r\}$. Let $u_0 \in U$ be nonzero and let $i \in \{1, ..., r\}$ such that $v_i^{(u_0)} \neq 0$. Then the map

$$\begin{array}{rccc} \pi_i : & U & \to & V_i \\ & u & \mapsto & v_i^{(u)} \end{array}$$

is a nonzero G-module homomorphism, so $\mathrm{Im}(\pi_i) = V_i$ and π_i is an isomorphism. □

The following proposition is a simple application of Maschke's theorem.

Proposition 16.2. *Let $\varphi : U \to V$ be a G-module homomorphism. Then $U = \ker(\varphi) \oplus W$ where $W \cong \mathrm{Im}(\varphi)$.*

Proof. By Maschke's theorem, $U = \ker(\varphi) \oplus W$ for some W. The map

$$\begin{array}{rccc} \varphi : & W & \to & \mathrm{Im}(\varphi) \\ & w & \mapsto & w\varphi \end{array}$$

is a G-module isomorphism. □

Proposition 16.1 applied to the G-module $\mathbb{C}G$ shows that if $\mathbb{C}G = V_1 \oplus ... \oplus V_r$ is a decomposition into irreducible modules, then any irreducible submodule $U \subseteq \mathbb{C}G$ is isomorphic to some V_i. The following theorem makes an even stronger statement, and shows that $\mathbb{C}G$ actually encodes every irreducible G-module in this way.

Theorem 16.1. *Let $\mathbb{C}G = V_1 \oplus ... \oplus V_r$ be a decomposition into irreducible G-modules. If U is any irreducible G-module, then $U \cong V_i$ for some i.*

Proof. Fix some nonzero $u_0 \in U$ and note that since $\{u_0 x \mid x \in \mathbb{C}G\} \subseteq U$ is a nontrivial submodule, $U = \{u_0 x \mid x \in \mathbb{C}G\}$. Therefore the map

$$\begin{array}{rccc} \varphi : & \mathbb{C}G & \to & U \\ & x & \mapsto & u_0 x \end{array}$$

is a surjective G-module homomorphism, so by the previous proposition $\mathbb{C}G \cong \ker(\varphi) \oplus W$ where $W \cong \mathrm{Im}(\varphi) = U$. Since W is a submodule of $\mathbb{C}G$, there exists some $i \in \{1, ..., r\}$ such that $U \cong W \cong V_i$. □

Corollary 16.1. *There are only finitely many irreducible submodules of G (over $\mathbb{C}$). In particular, there are at most* $\dim(\mathbb{C}G) = |G|$.

17 Schur's lemma revisited

Let V be a G module and let I index its (non-isomorphic) irreducible submodules V^λ, where $\lambda \in I$. Then we write

$$V = \bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus m_\lambda} = \underbrace{V^{\lambda_1} \oplus ... \oplus V^{\lambda_1}}_{m_{\lambda_1} \text{ times}} \oplus ... \oplus \underbrace{V^{\lambda_r} \oplus ... \oplus V^{\lambda_r}}_{m_{\lambda_r} \text{ times}}$$

If $\mathbb{C}G = \bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus m_\lambda}$, then what are the multiplicities m_λ? For abelian groups, we will see that $\mathbb{C}G = V_1 \oplus ... \oplus V_{|G|}$, in which case the multiplicities are all $m_\lambda = 1$.

Let U and V be G-modules. We define

$$\mathrm{Hom}_{\mathbb{C}G}(U, V) = \{\varphi : U \to V \mid \varphi \text{ is a } G\text{-module homomorphism}\}$$

$\mathrm{Hom}_{\mathbb{C}G}(U, V)$ is a vector space with

$$u(a\varphi + b\theta) = a(u\varphi) + b(u\theta)$$

for $a, b \in \mathbb{C}$, $u \in U$, and $\varphi, \theta \in \mathrm{Hom}_{\mathbb{C}G}(U, V)$. With this notation, we can restate Schur's lemma as follows:

Theorem 17.1. *(Schur's Lemma) Let U and V be irreducible G-modules. Then*

$$\dim\left(\mathrm{Hom}_{\mathbb{C}G}(U, V)\right) = \begin{cases} 1, & U \cong V \\ 0, & U \ncong V \end{cases}$$

Proof. If $U \ncong V$ then it is clear from our previous statement of Schur's lemma that $\mathrm{Hom}_{\mathbb{C}G}(U, V) = \{0\}$ and so $\dim\left(\mathrm{Hom}_{\mathbb{C}G}(U, V)\right) = 0$. Suppose $U \cong V$. Let $\varphi, \theta \in \mathrm{Hom}_{\mathbb{C}G}(U, V)$ be nonzero, so that both maps are isomorphisms and therefore invertible since U and V are irreducible. Then $\varphi\theta^{-1} : U \to U$ is a nonzero homomorphism so by Schur's lemma $\varphi\theta^{-1} = \lambda \cdot 1_U$ for some $\lambda \in \mathbb{C}$. In this case $\varphi = \lambda\theta$, so $\mathrm{Hom}_{\mathbb{C}G}(U, V) = \mathbb{C}\text{-span}\{\theta\}$ and $\dim\left(\mathrm{Hom}_{\mathbb{C}G}(U, V)\right) = 1$. □

With this statement of Schur's lemma, we can answer our previous question about the multiplicities m_λ with the following theorem.

Theorem 17.2. *Let $\left\{V^\lambda\right\}_{\lambda \in I}$ be the set of all non-isomorphic G-modules, and suppose*

$$V = \bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus m_\lambda}, \qquad \text{where } m_\lambda \in \mathbb{Z},\ m_\lambda \geq 0.$$

Then for each $\lambda \in I$, $m_\lambda = \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(V, V^\lambda\right)\right) = \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(V^\lambda, V\right)\right)$.

To prove this theorem, we require the following lemma.

Lemma 17.1. *Let U, V, and W be G-modules. Then*

$$\mathrm{Hom}_{\mathbb{C}G}\left(U \oplus W, V\right) \cong \mathrm{Hom}_{\mathbb{C}G}\left(U, V\right) \oplus \mathrm{Hom}_{\mathbb{C}G}\left(W, V\right)$$

$$\mathrm{Hom}_{\mathbb{C}G}\left(V, U \oplus W\right) \cong \mathrm{Hom}_{\mathbb{C}G}\left(V, U\right) \oplus \mathrm{Hom}_{\mathbb{C}G}\left(V, W\right)$$

as vector spaces.

Proof. The map

$$\begin{array}{lcl} \mathrm{Hom}_{\mathbb{C}G}\left(U \oplus W, V\right) & \to & \mathrm{Hom}_{\mathbb{C}G}\left(U, V\right) \oplus \mathrm{Hom}_{\mathbb{C}G}\left(W, V\right) \\ \left(\begin{array}{rccc} \varphi: & U \oplus W & \to & V \\ & u+w & \mapsto & u\varphi + w\varphi \end{array} \right) & \mapsto & \left(\begin{array}{rccc} \varphi_U: & U & \to & V \\ & u & \mapsto & u\varphi \end{array} \right) + \left(\begin{array}{rccc} \varphi_W: & W & \to & V \\ & w & \mapsto & w\varphi \end{array} \right) \end{array}$$

is a bijective linear transformation. Likewise, the map

$$\begin{array}{lcl} \mathrm{Hom}_{\mathbb{C}G}\left(V, U \oplus W\right) & \to & \mathrm{Hom}_{\mathbb{C}G}\left(V, U\right) \oplus \mathrm{Hom}_{\mathbb{C}G}\left(V, W\right) \\ \left(\begin{array}{rccc} \varphi: & V & \to & U \oplus W \\ & v & \mapsto & u_v + w_v \end{array} \right) & \mapsto & \left(\begin{array}{rccc} \varphi_U: & V & \to & U \\ & v & \mapsto & u_v \end{array} \right) + \left(\begin{array}{rccc} \varphi_W: & V & \to & W \\ & v & \mapsto & w_v \end{array} \right) \end{array}$$

is also a bijective linear transformation. □

We may now prove the theorem.

Proof. From the lemma, we can expand $\mathrm{Hom}_{\mathbb{C}G}\left(V, V^\lambda\right)$ into a direct sum, giving

$$\begin{aligned} \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(V, V^\lambda\right)\right) &= \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(\bigoplus_{\mu \in I} \left(V^\mu\right)^{\oplus m_\mu}, V^\lambda\right)\right) \\ &= \dim\left(\bigoplus_{\mu \in I} \mathrm{Hom}\left(V^\mu, V^\lambda\right)^{\oplus m_\mu}\right) \\ &= \sum_{\mu \in I} m_\mu \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(V^\mu, V^\lambda\right)\right) \\ &= m_\lambda. \end{aligned}$$

A parallel statement shows that $m_\lambda = \dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(V^\lambda, V\right)\right)$. □

With this theorem in hand, we may now make an even more explicit statement with regard to our decomposition of $\mathbb{C}G$ into G modules.

Theorem 17.3. *Let $\left\{V^\lambda\right\}_{\lambda \in I}$ be the set of non-isomorphic irreducible G-modules. Then*

$$\mathbb{C}G = \bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus \dim\left(V^\lambda\right)}.$$

Proof. From our previous results, we need only show that

$$\dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right)\right) = \dim\left(V^\lambda\right).$$

To this end let $V^\lambda = \mathbb{C}\text{-span}\{v_1, ..., v_d\}$, where $d = \dim\left(V^\lambda\right)$. Then the maps

$$\begin{array}{rccc} \varphi_i: & \mathbb{C}G & \to & V^\lambda \\ & x & \mapsto & v_i x \end{array}$$

are G-module homomorphisms, so $\varphi_i \in \mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right)$ for each $i \in \{1, ..., d\}$. We want to show that $\{\varphi_1, ..., \varphi_d\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right)$. For this, suppose $\theta \in \mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right)$ and $x \in \mathbb{C}G$. Then for some $c_i \in \mathbb{C}$,

$$x\theta = (1\theta)x = (c_1 v_1 + ... + c_d v_d)x = c_1 v_1 x + ... + c_d v_d x = c_1(x\varphi_1) + ... + c_d(x\varphi_d)$$

in which case $\theta = c_1\varphi_1 + ... + c_d\varphi_d$. We conclude that $\mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right) = \mathbb{C}\text{-span}\{\varphi_1, ..., \varphi_d\}$. Now suppose that for some $b_1, ..., b_d \in \mathbb{C}$,

$$b_1\varphi_1 + ... + b_d\varphi_d = 0.$$

In this case we have that

$$b_1(1\varphi_1) + ... + b_d(1\varphi_d) = b_1v_1 + ... + b_dv_d = 0$$

so $b_1 = ... = b_d = 0$ since $\{v_1, ..., v_d\}$ is a basis. Therefore $\{\varphi_1, ..., \varphi_d\}$ is a basis, so

$$\dim\left(\mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}G, V^\lambda\right)\right) = \dim\left(V^\lambda\right)$$

and

$$\mathbb{C}G = \bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus \dim\left(V^\lambda\right)}.$$

$\square$

Corollary 17.1. $|G| = \sum_{\lambda \in I} \dim\left(V^\lambda\right)^2$.

Proof. Note that $\dim(\mathbb{C}G) = |G|$, and that

$$\dim(\mathbb{C}G) = \dim\left(\bigoplus_{\lambda \in I} \left(V^\lambda\right)^{\oplus \dim\left(V^\lambda\right)}\right) = \sum_{\lambda \in I} \dim\left(V^\lambda\right)\dim\left(V^\lambda\right) = \sum_{\lambda \in I} \dim\left(V^\lambda\right)^2.$$

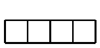
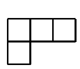
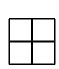
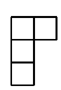
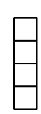
$\square$

18 Representations of the symmetric group

The representations of the symmetric group represent one of the great success stories in representation theory. Not only does the theory work out in a particularly beautiful way, but the technique used to decompose and classify the irreducible S_n-modules generalizes to a variety of other groups. In addition, the representations of the symmetric group yield some of the most accessible applications of representation theory to other disciplines such as chemistry. For these reasons we devote some time here to proving the classification theorem for the irreducible representations of S_n. For this we will need to introduce a few concepts borrowed from combinatorics.

A *partition* λ of n is a collected of boxes stacked into a corner. The parts $\lambda_1, ..., \lambda_k$ of λ are given by the number of boxes in the corresponding rows. Equivalently, a partition λ of n is a sequence of numbers $(\lambda_1, ..., \lambda_k)$ such that $|\lambda| = \lambda_1 + ... + \lambda_k = n$ and $\lambda_k \geq ... \geq \lambda_1 > 0$.

Example. *The partitions of* $n = 4$ *are given by*

 (4),  $(3,1)$,  $(2,2)$,  $(2,1,1)$,  $(1,1,1,1)$.

The *content* of a box $\square$ in the ith row and jth column of a partition is defined by

$$C(\square) = j - i.$$

Example. *The possible contents of a partition are given by*

0	1	2	3	...
−1	0	1	2	...
−2	−1	0	1	...
−3	−2	−1	0	...
⋮	⋮	⋮	⋮	⋱

A *standard tableau* T of shape λ is a filling of the boxes of λ by $1, ..., n$ such that the entries increase along the rows and columns.

Example. *The standard tableaux of shape* $\ydiagram{3,2}$ *are*

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau} \qquad \begin{ytableau} 1 & 3 & 4 \\ 2 & 4 \end{ytableau}$$

A natural question to ask is how many standard tableaux exist for a given partition λ. We will solve this combinatorial question later. Let T be a standard tableau, and define

$$C(T(i)) = \text{Content of the box } \square \text{ containing } i.$$

Example. *For the following standard tableau*

$$T = \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau}$$

we have

$$C(T(5)) = 0, \qquad C(T(4)) = 1, \qquad C(T(2)) = 1.$$

If $w \in S_n$ and T is a tableau with n boxes, let Tw denote the tableau formed by applying w to each of the boxes of T. Note that if T is a standard tableau, then Tw need not be. With this notation, we can now explicitly describe the irreducible S_n-modules.

Theorem 18.1. *For the symmetric group $S_n = \langle s_1, ..., s_{n-1} \rangle$,*

(1) The irreducible S_n-modules V^λ are indexed by partitions of n.

(2) $\dim\left(V^\lambda\right)$ is equal to the number of standard tableaux of shape λ.

(3) $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T$ is standard tableau of shape $\lambda\}$, and

$$v_T s_i = C_T(i) v_T + (1 + C_T(i))\, v_{Ts_i}$$

where $C_T(i) = [C(T(i+1)) - C(T(i))]^{-1}$ and $v_{Ts_i} = 0$ if Ts_i is not standard.

Example. *The irreducible S_4-modules are given by*

$$\begin{aligned}
V^{\ydiagram{4}} &= \mathbb{C}\text{-span}\{v_T \mid T = T_1\} \\
V^{\ydiagram{3,1}} &= \mathbb{C}\text{-span}\{v_T \mid T = T_{21}, T_{22}, T_{23}\} \\
V^{\ydiagram{2,2}} &= \mathbb{C}\text{-span}\{v_T \mid T = T_{31}, T_{32}\} \\
V^{\ydiagram{2,1,1}} &= \mathbb{C}\text{-span}\{v_T \mid T = T_{41}, T_{42}, T_{43}\} \\
V^{\ydiagram{1,1,1,1}} &= \mathbb{C}\text{-span}\{v_T \mid T = T_5\}
\end{aligned}$$

where

$$T_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$$

$$T_{21} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad T_{22} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T_{23} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$T_{31} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad T_{32} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

$$T_{41} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad T_{42} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad T_{43} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

$$T_5 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

We will prove this theorem with the following four lemmas:

(1) For each partition λ of n, V^λ is an S_n-module.

(2) For two partitions λ and μ of n, $V^\lambda \cong V^\mu$ if and only if $\lambda = \mu$.

(3) For each partition λ of n, V^λ is irreducible.

(4) $n! = \sum_{|\lambda|=n} \dim \left(V^\lambda\right)^2$.

Lemma 18.1. *For each partition λ of n, V^λ is an S_n-module.*

Proof. Let λ be a partition of n. By definition V^λ is a vector space, so we only need to check that for all $v_T \in V^\lambda$ and $g, h \in S_n$, $(v_T \circ g) \circ h = v_T \circ (gh)$ It suffices to check this on the set of generators $s_1, ..., s_{n-1}$, in which case we must verify the following relations:

(a) $(v_T s_i)\, s_i = v_T$ for all standard tableaux T of shape λ.

(b) $(v_T s_i)\, s_j = (v_T s_j)\, s_i$ for all T and $|i - j| > 1$.

(c) $((v_T s_i)\, s_{i+1})\, s_i = ((v_T s_{i+1})\, s_i)\, s_{i+1}$ for all T.

Let T be a standard tableau of shape λ. To check (a), note that i and $i+1$ can appear in T is three different ways:

(i) $i+1$ can appear directly below i, as below:

$$\begin{array}{|c|}\hline i \\ \hline i{+}1 \\ \hline\end{array}$$

In this case $C(T(i+1)) = C(T(i) - 1$ so $C_T(i) = -1$, and Ts_i is not standard so $v_{Ts_i} = 0$. As such

$$v_T s_i = C_T(i) v_T + (1 + C_T(i))\, v_{Ts_i} = -v_T$$

so $(v_T s_i)\, s_i = -v_T s_i = v_T$.

(ii) $i+1$ can appear directly to the right of i, as below:

$$\begin{array}{|c|c|}\hline i & i{+}1 \\ \hline\end{array}$$

Quite similar to (ii), here $C(T(i+1)) = C(T(i) + 1$ so $C_T(i) = 1$, and Ts_i is not standard so $v_{Ts_i} = 0$. Hence $v_T s_i = v_T$ so $(v_T s_i)\, s_i = v_T s_i = v_T$.

(iii) $i+1$ can appear in a different row and column than i, as below:

$$\begin{array}{cc} & \boxed{i{+}1} \\ \boxed{i} & \end{array}$$

In this case Ts_i is standard so $v_{(Ts_i)s_i} = T$, and $C_{Ts_i}(i) = -C_T(i)$. Therefore

$$\begin{aligned}
(v_T s_i) s_i &= \left(C_T(i) v_T + (1 + C_T(i)) v_{Ts_i}\right) s_i \\
&= C_T(i) v_T s_i + (1 + C_T(i))\, v_{Ts_i} s_i \\
&= C_T(i)\left(C_T(i) v_T + (1 + C_T(i)) v_{Ts_i}\right) + (1 + C_T(i))\left(-C_T(i) v_{Ts_i} + (1 - C_T(i)) v_T\right) \\
&= C_T(i)^2 v_T + \left(C_T(i) + C_T(i)^2\right) v_{Ts_i} - \left(C_T(i) + C_T(i)^2\right) v_{Ts_i} + \left(1 - C_T(i)^2\right) v_T \\
&= C_T(i) v_T.
\end{aligned}$$

To prove (b), note that $Ts_i s_j = Ts_j s_i$ for $|i - j| > 1$, so

$$\begin{aligned}
(v_T s_i)\, s_j &= C_T(i) V_T s_j + (1 + C_T(i))\, v_{Ts_i} s_j \\
&= C_T(i) C_T(j) \left(v_T + v_{Ts_i} + v_{Ts_j} + v_{Ts_i s_j}\right) + C_T(i)\left(v_{Ts_j} + v_{Ts_i s_j}\right) \\
&\quad + C_T(j)\left(v_{Ts_i} + v_{Ts_i s_j}\right) + v_{Ts_i s_j} \\
&= C_T(i) C_T(j) \left(v_T + v_{Ts_i} + v_{Ts_j} + v_{Ts_j s_i}\right) + C_T(i)\left(v_{Ts_j} + v_{Ts_j s_i}\right) \\
&\quad + C_T(j)\left(v_{Ts_i} + v_{Ts_j s_i}\right) + v_{Ts_j s_i} \\
&= C_T(j) V_T s_i + (1 + C_T(j))\, v_{Ts_j} s_i \\
&= (v_T s_j)\, s_i.
\end{aligned}$$

To check (c), we again note that there are essentially three ways in which $i, i+1, i+2$ can appear in T:

(i) i, $i+1$, and $i+2$ can appear all in the same row or column, as below:

$$\begin{array}{|c|}\hline i \\ \hline i{+}1 \\ \hline i{+}2 \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|c|}\hline i & i{+}1 & i{+}2 \\ \hline\end{array}$$

In both cases we have

$$\left((v_T s_i)\, s_{i+1}\right) s_i = \left(\pm v_T s_{i+1}\right) s_i = v_T s_i = \pm v_T = \left((v_T s_{i+1})\, s_i\right) s_{i+1}.$$

(ii) Two of $i, i+1, i+2$ can appear in the same row or column of T, with the third number appearing in a different row and column from the others, as below:

$$\begin{array}{ccc} & & \boxed{i{+}2} \\ \boxed{i} & \boxed{i{+}1} & \end{array} \quad \text{or} \quad \begin{array}{ccc} & & \boxed{i{+}1} \\ \boxed{i} & \boxed{i{+}2} & \end{array} \quad \text{or} \quad \begin{array}{ccc} & & \boxed{i} \\ \boxed{i{+}1} & \boxed{i{+}2} & \end{array}$$

By the same sort of calculations as above, it follows that $((v_T s_i)\, s_{i+1})\, s_i = ((v_T s_{i+1})\, s_i)\, s_{i+1}$ in each of these cases.

(iii) $i, i+1, i+2$ can each appear in different rows and columns, as below:

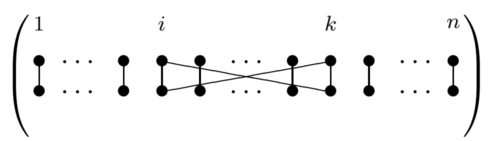
$$\begin{array}{ccc} & & \boxed{i{+}2} \\ & \boxed{i{+}1} & \\ \boxed{i} & & \end{array}$$

By an argument similar to that used in part (b), it follows in this case that $((v_T s_i)\, s_{i+1})\, s_i = ((v_T s_{i+1})\, s_i)\, s_{i+1}$.

Hence the generating relations of S_n hold for the given action of G on V^λ, so for all $g, h \in S_n$, $(v_T \circ g) \circ h = v_T \circ (gh)$. We conclude that V^λ is an S_n-module. $\square$

Before beginning on our next major lemma, we make a few definitions. In particular, we define the *Murphy elements* $\{m_k \mid 1 \leq k \leq n\} \subseteq \mathbb{C}S_n$, a special subset of the group algebra which will help us to show that $V^\lambda \cong V^\mu$ if and only if $\lambda = \mu$ for any two partitions λ, μ of n. Let $m_1 = 0$ and for $1 < k \leq n$ define

$$m_k = \sum_{i=1}^{k-1} (i\ k) = \sum_{i=1}^{k-1}$$



These elements have the very useful property outlined in the following lemma.

Lemma 18.2. *Let T be a standard tableau. Then $v_T m_k = C(T(k)) v_T$.*

Proof. We prove this statement by induction on k. Clearly $v_T m_1 = 0 = C(T(1)) v_T$ since in a standard tableau the number 1 is always in the top left box. Now assume the lemma holds for $r < k$, and note that $m_k = s_{k-1} m_{k-1} s_{k-1} + s_{k-1}$. Let $C_k = C(T(k))$. Then

$$\begin{aligned}
v_T m_k &= (v_T\,(s_{k-1} m_{k-1} + 1)\, s_{k-1}) \\
&= \left(\frac{1}{C_k - C_{k-1}} v_T m_{k-1} + \left(1 + \frac{1}{C_k - C_{k-1}}\right) v_{Ts_{k-1}} m_{k-1} + v_T \right) s_{k-1} \\
&= \left(\frac{C_{k-1}}{C_k - C_{k-1}} v_T + C_k \left(1 + \frac{1}{C_k - C_{k-1}}\right) v_{Ts_{k-1}} + v_T \right) s_{k-1} \\
&= \left(C_k \frac{1}{C_k - C_{k-1}} v_T + C_k \left(1 + \frac{1}{C_k - C_{k-1}}\right) v_{Ts_{k-1}} \right) s_{k-1} \\
&= C_k\, (v_T s_{k-1})\, s_{k-1} \\
&= C(T(k)) v_T.
\end{aligned}$$

This proves our inductive hypothesis for all k. $\square$

We may now prove our second lemma.

Lemma 18.3. *Let λ and μ be partitions of n. Then $V^\lambda \cong V^\mu$ if and only if $\lambda = \mu$.*

Proof. Let T be a standard tableau, and note that the Murphy elements applied to v_T give a sequence

$$(v_T m_1, v_T m_2, ..., v_T m_n) = (C(T(1)), C(T(2), ..., C(T(n)) v_T$$

The sequence $(C(T(1)), C(T(2), ..., C(T(n))$ completely determines T, since for each $i \in \{1, ..., n\}$, if we know which boxes contain $1, ..., i-1$ then $C(T(i))$ determines which box must contain i. Since we know that 1 is contained in the top left box, a simple inductive argument shows that $(C(T(1)), C(T(2), ..., C(T(n))$ tells us where each $i \leq n$ is located. Furthermore, v_T is the only vector in V^λ that gives rise to this sequence. If $\mu \neq \lambda$ then V^μ has no vector which corresponds to $(C(T(1)), C(T(2), ..., C(T(n))$ so $V^\lambda \not\cong V^\mu$. Conversely, if $\varphi : V^\lambda \to V^\mu$ is an isomorphism, then $(v_T\varphi)\, m_k = C(T(k))\, (v_T\varphi)$ for all $k \in \{1, ..., n\}$ for all standard tableaux T of shape λ, so $\lambda = \mu$. ◻

Our next task is to show that V^λ is irreducible.

Lemma 18.4. *Let λ be a partition of n. Then V^λ is irreducible.*

Proof. We prove this lemma with a series of claims. Let λ be a partition of n, and let

$$v_0 = \sum_Q a_Q v_Q \in V^\lambda$$

be a nonzero element of V^λ, where the sum ranges over all standard tableaux Q of shape λ and $a_Q \in \mathbb{C}$ for each such Q. Let

$$M = \{v_0 x \mid x \in \mathbb{C}S_n\}.$$

We first claim the following:

Claim. *If T is a standard tableau of shape λ and $a_T \neq 0$, then $v_T \in M$.*

To prove this claim, let T be a standard tableau of shape λ and define

$$\pi_T = \prod_{k=1}^{n} \prod_{j_k \in \{-n,...,n\} \setminus \{C(T(k))\}} \left(\frac{1}{C(T(k)) - j_k} (m_k - j_k I) \right) \in \mathbb{C}S_n.$$

Note that

$$v_Q \pi_T = \begin{cases} a_T v_T, & \text{if } Q = T \\ 0, & \text{if } Q \neq T \end{cases}$$

so if $a_T \neq 0$ then $v_0 \left(\frac{1}{a_T} \pi_T \right) = v_T \in M$. This proves our first claim, and our second claim goes as follows:

Claim. *If P, Q are two standard tableaux of shape λ, then there exists a sequence $i_1, i_2, ..., i_k \in \{1, ..., n-1\}$ such that the follow sequence of tableaux are all standard:*

$$Q, \qquad Qs_{i_1}, \qquad Qs_{i_1}s_{i_2}, \qquad \ldots, \qquad Qs_{i_1}s_{i_2}...s_{i_k} = P.$$

To prove this claim we demonstrate an algorithm by which such a sequence can be chosen. Let C be the column reading tableau of shape λ; that is, the tableau in which $i+1$ is directly below i unless i is the bottom entry in a column for each $i \in \{1, ..., n\}$. For example,

$$C = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}$$

is a column reading tableau. We will show how to choose two sequences $i_1, ..., i_k$ and $j_1, ..., j_l$ such that

$$Q, \qquad Qs_{i_1}, \qquad \ldots, \qquad Qs_{i_1}...s_{i_k} = C = Ps_{j_1}...s_{j_l}, \qquad \ldots, \qquad Ps_{j_1}, \qquad P$$

are all standard.

Let i_1 be the largest number in Q such that i_1+1 is strictly *south* and strictly *west* of i_1. If no such i_1 exists, then $Q = C$; otherwise Qs_{i_1} is standard. Choose i_2 from Qs_{i_1} is the same way, and inductively construct a sequence $i_1, ..., i_k$ such that Q, Qs_{i_1}, ... $Qs_{i_1}...s_{i_k}$ are all standard. By definition, in the terminating tableaux $Qs_{i_1}...s_{i_k}$ there is no i such that $i+1$ is strictly south and strictly west of i_1, so $Qs_{i_1}...s_{i_k} = C$. Choose a sequence $j_1, ..., j_l$ is the same way for P, so that P, Ps_{j_1}, ... $Ps_{j_1}...s_{j_l} = C$ are all standard. Then for the sequence $i_1, ..., i_k, j_l, ..., j_1$ we have that

$$Q, \qquad Qs_{i_1}, \qquad \ldots, \qquad Qs_{i_1}...s_{i_k} = Ps_{j_1}...s_{j_l}, \qquad \ldots, \qquad Qs_{i_1}...s_{i_k}s_{j_l}...s_{j_1} = P$$

are all standard. As an example of this process, consider

$$Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 7 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}$$

Here we have

$$Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 7 \\ \hline 5 \\ \cline{1-1}\end{array} \xrightarrow{s_4} \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 \\ \cline{1-1}\end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \hline 4 \\ \cline{1-1}\end{array} \xrightarrow{s_3} \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array} = C$$

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array} \xrightarrow{s_5} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array} \xrightarrow{s_4} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 \\ \cline{1-1}\end{array} \xrightarrow{s_5} \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \hline 4 \\ \cline{1-1}\end{array} \xrightarrow{s_3} \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array} = C$$

This proves our second claim. Our third claim is

Claim. *If Q and Qs_i are standard tableaux, then the coefficients of v_{Qs_i} in $v_Q s_i$ is nonzero.*

The proof of this claim follows by definition. Note that

$$v_Q s_i = C_Q(i) v_Q + (1 + C_Q(i))\, v_{Qs_i}.$$

If

$$1 + C_Q(i) = 1 + \frac{1}{C(Q(i+1)) - C(Q(i))} = 0,$$

then $C(Q(i+1)) = C(Q(i)) - 1 < C(Q(i))$ in which case Qs_i is not standard.

To prove the lemma as a whole, it suffices to show that $V^\lambda = M = \{v_0 x \mid x \in \mathbb{C}S_n\}$. This follows since if V^λ is reducible then there exists a proper nontrivial submodule U, in which case for any nonzero $v_0 \in U \subseteq V$, the set $M = \{v_0 x \mid x \in \mathbb{C}S_n\}$ is a submodule of U. If $M = V$ for all nonzero $v_0 \in V^\lambda$, then clearly there cannot exist any proper nontrivial submodules, so V^λ is irreducible.

To this end, recall that $v_0 = \sum_Q a_Q v_Q$ is nonzero. For some standard tableau T of shape λ, the coefficient $a_T \neq 0$ so by our first claim $v_T \in M$. We want to show that for every standard tableau P of shape λ, $v_P \in M$. Given an arbitrary standard tableau P of shape λ, there exists by our second claim a sequence $i_1, ..., i_k$ such that T, Ts_{i_1}, ... $Ts_{i_1}...s_{i_k} = P$ are all standard. It follows by our third claim that the coefficient of v_P is nonzero is $v_T s_{i_1}...s_{i_k} \in M$. Therefore by our first claim, $v_P \in M$. Hence $M = V^\lambda$ so V^λ is irreducible. $\square$

To show that the isomorphic, irreducible modules V^λ comprise all the irreducible S_n-modules, we want to prove that

$$\begin{aligned} |S_n| &= n! \\ &= \textstyle\sum_{|\lambda|=n} \dim\left(V^\lambda\right)^2 \\ &= \textstyle\sum_{|\lambda|=n} |\{\text{Standard tableaux of shape } \lambda\}|^2 \\ &= |\{\text{Pairs of standard tableaux of the same shape}\}| \end{aligned}$$

For this, we need a bijection

$$S_n \quad \leftrightarrow \quad \{\text{Pairs of standard tableaux of the same shape } \lambda\} \ .$$

We will obtain this bijection from the *RSK correspondence*, but first we need to introduce the algorithm known as *column insertion.*

Definition 13. *Let T be a standard tableau. If i is not contained in any of the boxes of T, the we can insert i into T to produce a new tableau, denoted*

$$\boxed{i} \rightarrow T$$

by the following algorithm:

(1) Let j be minimal in the first column of T such that $i < j$. If no such j exists, then place i in a new box at the bottom of the column. Otherwise, replace j with i.

(2) Insert j into the second column as in (1).

(3) Continue until no box is replaced by the preceding steps.

This algorithm is known as column insertion*.*

Example.

$$\boxed{4} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 9 \\ \hline 7 & 8 & \\ \cline{1-2} 10 & & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 8 \\ \hline 4 & 7 & 9 \\ \hline 10 & & \\ \cline{1-1} \end{array}$$

We now define the RSK correspondence, which describes a map between permutations and pairs of standard tableaux. We will later see that this map is a bijection.

Definition 14. *Let $w \in S_n$ and let (P_w, Q_w) be the least tableaux in the recursive sequence*

$$(\varnothing, \varnothing) = (P_0, Q_0), \qquad (P_1, Q_1), \qquad \ldots \qquad (P_n, Q_n) = (P_w, Q_w)$$

where for each $i \in \{1, ..., n\}$,

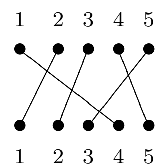
$$P_i \;=\; \boxed{iw} \;\rightarrow\; P_{i-1}$$

and Q_i is the tableau with the same shape as P_i and with i in the box not present in Q_{i-1}. The map

$$\begin{array}{rcl} S_n & \rightarrow & \{\textit{Pairs of standard tableaux with } n \textit{ boxes of the same shape}\} \\ w & \mapsto & (P_w, Q_w) \end{array}$$

is known as the RSK correspondence.

Example.



$$\mapsto \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right)$$

where

$$(\varnothing, \varnothing) \rightarrow \left(\begin{array}{|c|}\hline 4 \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \right)$$

$$\rightarrow \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right) \rightarrow \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right)$$

We can now prove our fourth lemma.

Lemma 18.5. *The RSK correspondence is a bijection. Equivalently,*

$$n! = \sum_{|\lambda|=n} \dim \left(V^\lambda\right)^2.$$

Proof. Since we understand the algorithms involved, the proof of this result amounts to checking that we can generate an arbitrary pair of tableaux of the same shape from a permutation $w \in S_n$, and then ensuring that distinct permutations generate distinct sequences of the form given in Definition 14. We omit these steps here. $\square$

These four lemmas prove our main theorem, which states that for the symmetric group $S_n = \langle s_1, ..., s_{n-1} \rangle$,

(1) The irreducible S_n-modules V^λ are indexed by partitions of n.

(2) $\dim \left(V^\lambda\right)$ is equal to the number of standard tableaux of shape λ.

(3) $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ is standard tableau of shape } \lambda\}$, and

$$v_T s_i = C_T(i) v_T + (1 + C_T(i))\, v_{Ts_i}$$

where $C_T(i) = [C(T(i+1)) - C(T(i))]^{-1}$ and $v_{Ts_i} = 0$ if Ts_i is not standard.

This completes our discussion of the representations of the symmetric group. Our next topic is character theory.