Math 8174: Homework 3

Due February 18–20, 2009

1. Suppose $G_1, G_2, \ldots, G_r$ are the irreducible G-modules. The *degree sequence* of a group G is the sequence $(\dim(G_1), \dim(G_2), \ldots, \dim(G_r))$. Suppose $|G| = 20$.

   (a) Show that G has less than 10 possible degree sequences.

   (b) It turns out that the $\dim(G_i)$ divides $|G|$ (we will hopefully show this later in the class). Using this fact, show that G must have at least 4 different one-dimensional modules.

2. Let $Z(G)$ denote the center of G.

   (a) Show that if G has a faithful irreducible representation (an injective irreducible representation), then $Z(G)$ is cyclic.

   (b) Is it generally true that the center of $\mathbb{C}G$ is $\mathbb{C}Z(G)$?

3. Let $n \in \mathbb{Z}_{\geq 1}$. Let λ be a partition of n. Let
$$V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ a standard tableau of shape } \lambda\}.$$
   For $i \in \{1, 2, \ldots, n-1\}$, define
$$s_i v_T = \begin{cases} v_{s_i(T)}, & \text{if } s_i(T) \text{ is standard,} \\ v_T, & \text{otherwise.} \end{cases}$$
   Is V^λ an S_n-module under this action? If so, what is its decomposition into irreducibles?

4. The natural module of S_n is the vector space
$$V = \mathbb{C}\text{-span}\{e_1, e_2, \ldots, e_n\},$$
   with the action
$$\begin{array}{ccc} S_n \times V & \longrightarrow & V \\ (w, e_i) & \mapsto & e_{w(i)} \end{array}$$
   Decompose V into irreducible S_n-modules.

5. Given a vector space V, let
$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ terms}}.$$
   The (infinite) group $\mathrm{GL}_n(\mathbb{C})$ acts on $(\mathbb{C}^n)^{\otimes r}$ by
$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n, \qquad \text{for } g \in \mathrm{GL}_n(\mathbb{C}),\ v_i \in \mathbb{C}^n.$$
   It turns out that for $r \leq n$,
$$\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}((\mathbb{C}^n)^r) \cong \mathbb{C}S_r,$$
   where
$$w(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{w(1)} \otimes v_{w(2)} \otimes \cdots \otimes v_{w(r)}, \qquad \text{for } w \in S_r,\ v_i \in \mathbb{C}^n.$$
   Show that for $n > 2$,
$$\mathrm{End}_{\mathrm{GL}_n(\mathbb{C})}((\mathbb{C}^n)^2) \cong \mathbb{C}S_2.$$
   Hint: One can use an elementary argument using heavily the subgroup $G(2, 1, n) \subseteq \mathrm{GL}_n(\mathbb{C})$.