Math 8174: Homework 3
Due February 18–20, 2009

1. Suppose $G_1, G_2, \ldots, G_r$ are the irreducible $G$-modules. The degree sequence of a group $G$ is the sequence $(\dim(G_1), \dim(G_2), \ldots, \dim(G_r))$. Suppose $|G| = 20$.
   (a) Show that $G$ has less than 10 possible degree sequences.
   (b) It turns out that the $\dim(G_i)$ divides $|G|$ (we will hopefully show this later in the class). Using this fact, show that $G$ must have at least 4 different one-dimensional modules.

2. Let $Z(G)$ denote the center of $G$.
   (a) Show that if $G$ has a faithful irreducible representation (an injective irreducible representation), then $Z(G)$ is cyclic.
   (b) Is it generally true that the center of $CG$ is $\mathbb{C}Z(G)$?

3. Let $n \in \mathbb{Z}_{\geq 1}$. Let $\lambda$ be a partition of $n$. Let $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ a standard tableau of shape } \lambda\}$.
   For $i \in \{1, 2, \ldots, n-1\}$, define
   $$s_i v_T = \begin{cases} v_{s_i(T)}, & \text{if } s_i(T) \text{ is standard,} \\ v_T, & \text{otherwise.} \end{cases}$$
   Is $V^\lambda$ an $S_n$-module under this action? If so, what is its decomposition into irreducibles?

4. The natural module of $S_n$ is the vector space $V = \mathbb{C}\text{-span}\{e_1, e_2, \ldots, e_n\}$, with the action
   $$S_n \times V \rightarrow V \quad (w, e_i) \mapsto e_{w(i)}$$
   Decompose $V$ into irreducible $S_n$-modules.

5. Given a vector space $V$, let $V^\otimes k = V \otimes V \otimes \cdots \otimes V$.
   The (infinite) group $\text{GL}_n(\mathbb{C})$ acts on $(\mathbb{C}^n)^\otimes r$ by
   $$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n, \quad \text{for } g \in \text{GL}_n(\mathbb{C}), \ v_i \in \mathbb{C}^n.$$
   It turns out that for $r \leq n$,
   $$\text{End}_{\text{GL}_n(\mathbb{C})}((\mathbb{C}^n)^\otimes r) \cong \mathbb{C}S_r,$$
   where
   $$w(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{w(1)} \otimes v_{w(2)} \otimes \cdots \otimes v_{w(r)}, \quad \text{for } w \in S_r, \ v_i \in \mathbb{C}^n.$$
   Show that for $n > 2$,
   $$\text{End}_{\text{GL}_n(\mathbb{C})}((\mathbb{C}^n)^2) \cong \mathbb{C}S_2.$$
   Hint: One can use an elementary argument using heavily the subgroup $G(2,1,n) \subseteq \text{GL}_n(\mathbb{C})$. 